

THE HIGHER TWISTED INDEX THEOREM FOR FOLIATIONS

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ABSTRACT. Given a gerbe L , on the holonomy groupoid \mathcal{G} of the foliation (M, \mathcal{F}) , whose pull-back to M is torsion, we construct a Connes Φ -map from the twisted Dupont-Sullivan bicomplex of \mathcal{G} to the cyclic complex of the L -projective leafwise smoothing operators on (M, \mathcal{F}) . Our construction allows to couple the K -theory analytic indices of L -projective leafwise elliptic operators with the twisted cohomology of $B\mathcal{G}$ producing scalar higher invariants. Finally by adapting the Bismut-Quillen superconnection approach, we compute these higher twisted indices as integrals over the ambient manifold of the expected twisted characteristic classes.

INTRODUCTION

The Atiyah-Singer families index theorem has been extended by Mathai, Melrose and Singer in [45, 46] to the class of projective families of operators. Given a smooth fibration $\pi : M \rightarrow B$ of closed manifolds, the authors of [45] consider local families of pseudodifferential operators over open sets of a trivializing cover of B such that the usual compatibility condition on triple overlaps fails by a scalar factor. This yields an integral 3-cohomology class of B which is known to correspond to a Dixmier-Douady (DD) class of a gerbe. A prototype example of such family arises from the attempt to define a Dirac operator along the fibers of some non K -oriented fibrations. Main geometric examples are though given by usual Dirac-type operators along the fibers with coefficients in auxiliary (horizontally) twisted vector bundles over the total manifold M . See also [55], where a closely related index theorem was proved as well. The proof given in [45] of the projective index theorem is the generalization to the twisted K -theory of the classical constructions of Atiyah-Singer for the families index theorem and they assumed the gerbe on the base manifold to be torsion. Later on, in [46], the projective index theorem was extended to deal with more general gerbes, called decomposable. In [4], the Bismut superconnection approach to the families index theorem [8, 56] allowed to give a new proof of the index formula, for Dirac-type operators with coefficients in twisted vector bundles, under the assumption that only the pull-back of the gerbe to M is torsion.

In this paper we extend the previous results and in particular those of [4] to projective operators along the leaves of foliations. In the non-twisted case, the index problem for elliptic operators along the leaves of a smooth foliation on a closed manifold, was solved in the seminal works of Connes [20, 21] and then of Connes-Skandalis [27]. In [21] and then in

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[27], Connes and Connes-Skandalis proved the index theorem for leafwise elliptic operators, an equality in the K -theory of the holonomy groupoid C^* -algebra. When the foliation is a fibration, the Connes-Skandalis theorem reduces to the Atiyah-Singer theorem for families [2], and the formula then lives in the K -group of the base manifold, so its Chern character is totally computable in terms of characteristic classes. For general foliations though, the K -group of the holonomy groupoid C^* -algebra is harder to compute, and many fundamental noncommutative geometry constructions were developed in order to provide computable pairings with such K -groups. The de Rham homology of the base manifold must now be replaced by the cyclic cohomology of some dense subalgebra of the groupoid C^* -algebra. An interesting and rich enough class of cyclic cocycles arises from the homology of the classifying space of the holonomy groupoid [25, 36, 37]. Recall for instance the breakthrough results of [22] which insure the extension of the pairing with the so-called fundamental class to the C^* -algebra K -theory, with deep topological consequences (see again [22]). The index formulae obtained using the full homology of the classifying space $B\mathcal{G}$ cannot in general be deduced from the Connes-Skandalis theorem since the pairings are only defined at the level of smooth dense subalgebras [36, 37, 18], and most of the higher index formulae are then tackled independently by using the superconnection formalism of Quillen [56] and by extending to foliations the classical approach to the families index problem [8, 6].

In the present paper, we use this superconnection approach in the context of foliations where the Dirac-type operator along the leaves acts on a vector bundle, which is twisted by a gerbe on the (reduced) holonomy groupoid. This groupoid plays in noncommutative geometry the role of the singular space of leaves. We also introduce many constructions which are valid for all étale Hausdorff Lie groupoids \mathcal{G} and for all \mathcal{G} -equivariant smooth submersions $\pi : P \rightarrow B$. Roughly speaking, a gerbe L on \mathcal{G} or a \mathcal{G} -gerbe, is provided by a triple $(\mathcal{L}, \varsigma, \nu)$ where \mathcal{L} is a gerbe on the unit manifold $\mathcal{G}^{(0)}$, ς is a \mathcal{G} -action, that is a line bundle over the arrows $\mathcal{G}^{(1)}$, with the natural compatibility condition in terms of a 2-morphism ν over the 2-arrows, see Definition 3.1. Our definition then agrees with Morita equivalence and with the approach in [58]. The space of compactly supported smooth sections of ς over $\mathcal{G}^{(1)}$ is then endowed with the structure of an involutive convolution algebra by using the 2-morphism ν . If we complete it with respect to the regular representations, we obtain the regular L -twisted C^* -algebra $C_r^*(\mathcal{G}, L)$ associated with the groupoid \mathcal{G} and the \mathcal{G} -gerbe L . In the case of foliations, this latter C^* -algebra is the L -twisted C^* -algebra of the foliation (M, \mathcal{F}) and it is the natural receptacle for indices of L -projective leafwise elliptic pseudodifferential operators, exactly as in the untwisted case [20]. On the other hand, the DD-class associated with any \mathcal{G} -gerbe L lives in the integral 3-cohomology $H^3(\mathcal{G}, \mathbb{Z})$ associated with the Lie groupoid \mathcal{G} , or equivalently in the 3-cohomology of its classifying space $B\mathcal{G}$. We give a construction of the Chern-Weil representative of this DD-class in the Dupont-Sullivan complex of compatible forms. The complex of the compatible forms was first introduced by D. Sullivan in [57] and applied to the study of classifying spaces and characteristic classes by J. Dupont, cf. [31]. When pulled-back to the lifted groupoid $P \rtimes \mathcal{G}$ with units P , we get a $P \rtimes \mathcal{G}$ -gerbe, whose DD-class is the pull-back $\pi^*[L]$ of $[L]$. In the case of foliations, the \mathcal{G} -submersion is free and proper and the pull-back DD-class $\pi^*[L]$ in $H^3(P \rtimes \mathcal{G}, \mathbb{Z}) \simeq H^3(M, \mathbb{Z})$ is also the pull-back of the DD-class on $B\mathcal{G}$ under the classifying map of our foliation. So we

naturally ask that this class is torsion. This corresponds for fibrations to asking that the DD-class pulled-back to the total manifold of the fibration is torsion and we recover the usual assumption, see [45, 46, 4]. Now for the free and proper \mathcal{G} -equivariant submersion $\pi : P \rightarrow B = \mathcal{G}^{(0)}$, L -twisted vector bundles on $M = P/\mathcal{G}$, are the $\pi^*\mathcal{L}$ -twisted bundles over P which are \mathcal{G} -equivariant, see Definition 4.1. Given such bundle $\mathcal{E} \rightarrow P$, the algebra $\Psi_{\pi^*\mathcal{L}}(P|\mathcal{G}^{(0)}, \mathcal{E})^{\mathcal{G}}$ of L -projective leafwise pseudodifferential operators is defined as the algebra of (properly supported) fiberwise $\pi^*\mathcal{L}$ -projective pseudodifferential operators which are \mathcal{G} -equivariant. These are the L -projective leafwise pseudodifferential operators on the foliation \mathcal{F} of M obtained by moding out the fibers of π and it can suggestively also be denoted $\Psi_L(M, \mathcal{F}; \mathcal{E})$. Its two-sided ideal $\Psi_L^{-\infty}(M, \mathcal{F}; \mathcal{E})$ of L -projective leafwise smoothing operators is then $\Psi_{\pi^*\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E})^{\mathcal{G}}$.

Leafwise ellipticity of an element A then insures the existence of a K -theory analytic index class $\text{ind}_a(A)$ which lives in the K -theory group $K(\Psi_L^{-\infty}(M, \mathcal{F}; \mathcal{E}))$ and hence a C^* -index $\text{Ind}_a(A)$ which lives in the K -theory of its C^* -completion, i.e.

$$\text{ind}_a(A) \in K(\Psi_L^{-\infty}(M, \mathcal{F}; \mathcal{E})) \text{ and } \text{Ind}_a(A) \in K(\overline{\Psi_L^{-\infty}(M, \mathcal{F}; \mathcal{E})}).$$

As for the non twisted case, the index class yields an analytic class in the K -theory of the gerbe C^* -algebra $C_r^*(\mathcal{G}, L)$ which is compatible with the Morita equivalence

$$C_r^*(\mathcal{G}, L) \sim \overline{\Psi_L^{-\infty}(M, \mathcal{F}; \mathcal{E})}$$

If we denote by \mathcal{A} the \mathcal{G} -equivariant Azumaya C^* -algebra bundle associated with \mathcal{E} then the principal symbol of any L -twisted leafwise elliptic operator defines a class

$$[\sigma(A)] \in K(C_0(T^*P|\mathcal{G}^{(0)}, \mathcal{A})^{\mathcal{G}}),$$

and we obtain in this way a well defined L -projective leafwise index map

$$\text{Ind}_a : K(C_0(T^*P|\mathcal{G}^{(0)}, \mathcal{A})^{\mathcal{G}}) \longrightarrow K(C_r^*(\mathcal{G}, L)).$$

The goal here is to define and compute the higher indices of L -projective leafwise elliptic operators associated with the pairings against the (periodized) twisted homology classes of the classifying space $B\mathcal{G}$. Our approach uses superconnections and yields directly to a topological formula in terms of twisted characteristic classes. More precisely, we define the pairing of the smooth index class $\text{ind}_a(A)$ with cyclic cocycles arising from the twisted Dupont-Sullivan complex of smooth currents $(\mathcal{C}_{\Delta}^{\bullet}(\mathcal{G})[(u^*)^{-1}, u^*], d_{\Omega(L)}^t)$, where u^* is a formal variable of degree +2. These currents correspond to differential forms which incorporate an orientation so that they can be integrated even on nonoriented manifolds. We achieve this pairing by constructing a morphism $\Lambda^{\mathcal{E}}$ from this Dupont-Sullivan complex to the (periodic) cyclic cohomology of the ideal $\Psi_{\pi^*\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E})^{\mathcal{G}}$, representing the morphism

$$\Lambda^{\mathcal{E}} : H_{\Omega(L)}(B\mathcal{G}) \longrightarrow \text{HC}^{\bullet}(\Psi_L^{-\infty}(M, \mathcal{F}; \mathcal{E})).$$

When the L -projective bundle \mathcal{E} is \mathbb{Z}_2 -graded by the involution γ , we obtain in the same way the corresponding morphism $\Lambda^{\mathcal{E}, \gamma}$. We thus obtain the geometric construction of cyclic cocycles we use, and of higher indices of any L -projective leafwise elliptic operator D acting

from \mathcal{E}^+ to \mathcal{E}^- associated with a given even degree cocycle c in the Dupont-Sullivan complex. The precise formula is:

$$\text{ind}_c(D) := \langle (u^*)^{-\frac{\deg c}{2}} \Lambda^{\mathcal{E}, \gamma}(c), \text{ind}_a D \rangle.$$

The pairing in the RHS is the usual normalized pairing between K -theory and cyclic cohomology.

Our morphism $\Lambda^{\mathcal{E}}$ relies on a twisted and simplicial extension of the so-called Connes Φ -map (see Section 2.8 in [25]).

Theorem 5.3 [Twisted Connes' Φ -map] *There exists a morphism of complexes*

$$\Phi^{\mathcal{E}}: (C_{\Delta}^{\bullet}(\mathcal{G})[(u^*)^{-1}, u^*], d_{\Omega(L)}^t) \longrightarrow C^{\bullet}(\mathcal{G}, \text{CC}^{\bullet}(\Psi_{\pi^* \mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}))),$$

which extends to the twisted case, the classical Connes Φ -map.

Recall that the classical map Φ has been introduced in [22, 25] as a map from the cohomology of $B\mathcal{G}$ with the coefficients in the orientation bundle to the cyclic cohomology of the smooth convolution algebra $C_c^{\infty}(\mathcal{G})$. This map is an injection, cf. [25], Section 2.8. In [15] Brylinski-Nistor computed the cohomology of $C_c^{\infty}(\mathcal{G})$ for a general Hausdorff étale \mathcal{G} . This computation has been extended to the non-Hausdorff case by Crainic in [28]. Our theorem above extends Connes' construction to the twisted case.

Then using constructions again from [15, 28] and the isomorphism

$$\text{HC}^{\bullet}(\Psi_{\pi^* \mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}) \rtimes \mathcal{G}) \longrightarrow \text{HC}^{\bullet}(\Psi_{\pi^* \mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E})^{\mathcal{G}})$$

the allowed morphism $\Lambda^{\mathcal{E}}$ is deduced (see Equation 5.2.4).

As explained above, we also include the computation of the higher index $\text{ind}_c(D)$ by a topological formula *à la Atiyah-Singer*. We concentrate on Dirac-type operators along the leaves twisted by an L -projective hermitian bundle. The topological formula that we obtain is, as expected, some pairing of the class of c with the twisted Atiyah-Singer characteristic class. In order to pair the class of c with cohomology classes on $M = P/\mathcal{G}$, we pull them back to the groupoid $\mathcal{P} := P \rtimes \mathcal{G}$ and use the Dupont-Sullivan complex of forms on \mathcal{P} and the well-defined cap product

$$(\Omega_{\bullet}^{\Delta}(\mathcal{P}), d_{\Omega(L)}) \otimes (C_{\Delta}^{\bullet}(\mathcal{P})[(u^*)^{-1}, u^*], d_{\Omega(L)}^t) \longrightarrow (C_{\Delta}^{\bullet}(\mathcal{P})[(u^*)^{-1}, u^*], d).$$

Recall that the pull-back of L to \mathcal{P} is torsion so that the Chern-Weil representative of its DD-class in the Dupont-Sullivan complex of \mathcal{P} is exact. Since the Lie groupoid \mathcal{P} is proper, free and cocompact, the twisted (and non-twisted) cohomology of \mathcal{P} is isomorphic to the cohomology of the closed manifold M . More precisely, the \hat{A} -form of the tangent bundle to the foliation \mathcal{F} is easily represented by a closed form $\hat{A}(P|\mathcal{G}^{(0)})$ in the non-twisted Dupont-Sullivan complex, while the Chern character of the L -projective bundle \mathcal{E} , with respect to the spinor bundle \mathcal{S} as in the notations of [7], is naturally represented by a closed form $\text{Ch}_L(\mathcal{E}/S)$ in the $\pi^* \Omega(L)$ -twisted Dupont-Sullivan complex of forms. Hence, we end up with a $d_{\pi^* \Omega(L)}$ -closed form $\hat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S)$ which can be paired with the $d_{\pi^* \Omega(L)}^t$ -closed smooth current $\pi^* c$ to yield a smooth closed current (for the non-twisted differential)

$$(\pi^* c) \cap \left(\hat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right) \in C_{\Delta}^{\bullet}(\mathcal{P})[(u^*)^{-1}, u^*].$$

Simplicial integration and integration over $M = P/\mathcal{G}$ allows to integrate the closed smooth current $(\pi^*c) \cap \left(\widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right)$ and extract eventually a well defined complex number:

$$\left\langle \pi^*c, \widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right\rangle.$$

Our main result reads:

Theorem 6.12. *Let c be a cocycle of even degree in $(\mathcal{C}_\Delta^\bullet(\mathcal{G})[(u^*)^{-1}, u^*], d_{\Omega(L)}^t)$.*

$$\text{ind}_c(D) = \left\langle \pi^*c, \widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right\rangle.$$

Notice that the cocycle c represents a class $[c]$ in $H_{\Omega(L)}(B\mathcal{G})$ and if $f : M \rightarrow B\mathcal{G}$ is a classifying map for our foliation \mathcal{F} on M , then $f^*[c] = [\pi^*c]$. An explicit example illustrating the geometric set up underlying Theorem 6.12 is given in Subsection 4.3.

Our theorem unifies several previous results. First of all in the non-twisted case, we recover the results of [37, 25] as well as the results of [5]. In the case of a fibration, the gerbe is defined on the base of the fibration and we recover the results of [45, 46, 4]. In the case of a covering manifold, this result has been proved by Mathai-Marcolli ([43], [44]), extending Connes-Moscovici higher index theorem [26] to the case of projective actions.

Let us recall that the twisted version of the K -theoretic Connes-Skandalis index theorem has been recently investigated by Carrillo Rouse and Wang ([19]) by using the Connes tangent groupoid approach. More precisely, [19] used a geometric set up à la Hilsum-Skandalis for the definition of twisted groupoid which is well adapted to the proof of twisted K -theoretic index theorems. Their geometric set up is equivalent to ours. Their focus however is on the K -theoretic results. The focus of this paper, on the other hand, is on cohomological index theorems in suitable cyclic cohomological groups and on explicit cohomological index formula.

Finally, we mention that Carey and Wang ([16]) have proven a Riemann-Roch theorem in twisted K -theory and used it to prove a cohomological formula for the index pairing in twisted K -theory, so as to get applications to D-branes and D-charges. See also the paper [17] for definition and properties of differential twisted K -theory.

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CONTENTS

Introduction	1
1. Homologies of Étale Lie groupoids	6

1.1. The de Rham bicomplex	7
1.2. The Dupont-Sullivan bicomplex	8
2. Gerbes and twisted bundles	10
2.1. Gerbes and connections	10
2.2. Twisted and transversally twisted bundles	14
2.3. Projective families of pseudodifferential operators	19
2.4. Connections and curvatures	23
3. Gerbes on groupoids	24
3.1. \mathcal{G} -Gerbes and their morphisms	24
3.2. Connections on \mathcal{G} -gerbes	25
3.3. DD class in the Dupont-Sullivan bicomplex	26
4. Projective \mathcal{G} -families and the analytic index	29
4.1. Twisted vector bundles on groupoids	29
4.2. Projective \mathcal{G} -operators	31
4.3. An example of twisted \mathcal{G} -family	32
4.4. Invariant operators and cross-products	33
4.5. Analytic index of projective \mathcal{G} -operators	34
5. The twisted Connes map	35
5.1. Review of cyclic homology	35
5.2. The Connes map	36
6. The higher twisted index formula	39
6.1. Projective superconnections on fibrations	39
6.2. Projective superconnections on foliations	41
6.3. Proof of the higher index theorem	46
Appendix A. Twisted cohomology	47
Appendix B. Review of the map Φ	49
Appendix C. Some JLO estimates for complete manifolds	50
References	51

1. HOMOLOGIES OF ÉTALE LIE GROUPOIDS

Let \mathcal{G} be an étale Hausdorff locally compact groupoid with units $\mathcal{G}^{(0)}$ and denote for $k \geq 1$ by $\mathcal{G}^{(k)}$ the set of composable k -tuples, i.e.

$$\mathcal{G}^{(k)} = \{(\gamma_1, \dots, \gamma_k) \in \mathcal{G}^k \text{ such that } r(\gamma_{i+1}) = s(\gamma_i), i = 1, \dots, k-1\}$$

We define for $l \geq 1$ and $0 \leq i_1 < \dots < i_{l+1} \leq k$, the maps $\text{pr}_{i_1 i_2 \dots i_{l+1}}^k : \mathcal{G}^{(k)} \rightarrow \mathcal{G}^{(l)}$ by

$$(1.0.1) \quad \text{pr}_{i_1 i_2 \dots i_{l+1}}^k(\gamma_1, \gamma_2, \dots, \gamma_k) := (\gamma_{i_1+1} \dots \gamma_{i_2}, \gamma_{i_2+1} \dots \gamma_{i_3}, \dots, \gamma_{i_l+1} \dots \gamma_{i_{l+1}}).$$

For $l = 1$ for instance, we have

$$\text{pr}_{ij}^k(\gamma_1, \dots, \gamma_k) := \gamma_{i+1} \dots \gamma_j, \quad \text{for } 0 \leq i < j \leq k.$$

For $l = 0$ we define the maps $\text{pr}_i^k : \mathcal{G}^{(k)} \rightarrow \mathcal{G}^{(0)}$ by

$$\text{pr}_i^k(\gamma_1, \gamma_2, \dots, \gamma_k) := s(\gamma_i) = r(\gamma_{i+1}).$$

By $\mathcal{G}^{(\bullet)}$ we denote the simplicial space defined as follows. The face maps are given for $k \geq 2$ by

$$(1.0.2) \quad \partial_i(\gamma_1, \dots, \gamma_k) = \begin{cases} (\gamma_2, \dots, \gamma_k) & \text{for } i = 0 \\ (\gamma_1, \dots, \gamma_i \gamma_{i+1}, \dots, \gamma_k) & \text{for } 0 < i < k \\ (\gamma_1, \dots, \gamma_{k-1}) & \text{for } i = k \end{cases}$$

and for $k = 1$, we define $\partial_0(\gamma_1) = r(\gamma_1)$ and $\partial_1(\gamma_1) = s(\gamma_1)$.

We denote as well by r the map

$$r = \text{pr}_0^k : \mathcal{G}^{(k)} \longrightarrow \mathcal{G}^{(0)} \text{ given by } \text{pr}_0^k(\gamma_1, \dots, \gamma_k) = r(\gamma_1).$$

We shall only be interested in (Hausdorff étale) Lie groupoids which we assume from now on, hence all the groupoid operations are smooth.

1.1. The de Rham bicomplex. We recall now the de Rham bi-complex $\Omega^\bullet(\mathcal{G}^{(\bullet)}, \mathbb{R})$ associated with the nerve of the groupoid \mathcal{G} and which computes the cohomology (with real coefficients) of its classifying space $B\mathcal{G}$. So,

$$\Omega^m(\mathcal{G}^{(k)}, \mathbb{R}) := C_c^\infty(\mathcal{G}^{(k)}, r^* \Lambda^m(T^* \mathcal{G}^{(0)})),$$

and if d is the de Rham differential on the manifold $\mathcal{G}^{(0)}$ then it lifts to the differential r^*d which will still be denoted d for simplicity. So, the differentials are

$$d : \Omega^\bullet(\mathcal{G}^{(\bullet)}, \mathbb{R}) \rightarrow \Omega^{\bullet+1}(\mathcal{G}^{(\bullet)}, \mathbb{R}) \text{ and } \partial : \Omega^\bullet(\mathcal{G}^{(\bullet)}, \mathbb{R}) \rightarrow \Omega^\bullet(\mathcal{G}^{(\bullet+1)}, \mathbb{R})$$

where ∂ is induced by the simplicial maps $\partial_i : \mathcal{G}^{(k)} \rightarrow \mathcal{G}^{(k-1)}$ given in (1.0.2). More precisely,

$$\partial(\omega) = (-1)^m \sum (-1)^i \partial_i^* \omega \text{ for } \omega \in \Omega^m(\mathcal{G}^{(\bullet)}).$$

Notice that $\partial_1^* \omega$ uses the bundle transformations $\gamma : \Lambda^\bullet(T_{s(\gamma)}^* \mathcal{G}^{(0)}) \rightarrow \Lambda^\bullet(T_{r(\gamma)}^* \mathcal{G}^{(0)})$ which are induced by the corresponding usual bundle maps $T_{r(\gamma)} \mathcal{G}^{(0)} \rightarrow T_{s(\gamma)} \mathcal{G}^{(0)}$, these latter being well defined for any γ since \mathcal{G} is étale.

Lemma 1.1. *Given a Hausdorff étale groupoid $\widehat{\mathcal{G}}$ which is proper, the cohomology of the complex $(\Omega^m(\widehat{\mathcal{G}}^{(\bullet)}), \partial)$ is trivial in positive degrees.*

Proof. Since $\widehat{\mathcal{G}}$ is proper and Hausdorff there exists $h \in C_c^\infty(\widehat{\mathcal{G}}^{(0)})$ such that

$$\forall z \in \widehat{\mathcal{G}}^{(0)}, \quad \sum_{\{g \in \widehat{\mathcal{G}} \mid s(g)=z\}} h(zg) = 1.$$

Recall that we have for $m \geq 1$ the isomorphisms

$$\gamma : \Lambda^m T_{s(\gamma)}^* \widehat{\mathcal{G}} \longrightarrow \Lambda^m T_{r(\gamma)}^* \widehat{\mathcal{G}},$$

with the usual functorial properties. Define then the operator $K : \Omega^m(\widehat{\mathcal{G}}^{(\bullet)}) \rightarrow \Omega^m(\widehat{\mathcal{G}}^{(\bullet-1)})$ by

$$K\omega(\gamma_1, \dots, \gamma_k) := \sum_{s(\gamma)=r(\gamma_1)} h(r(\gamma)) \times \gamma^{-1} \omega(\gamma, \gamma_1, \dots, \gamma_k).$$

A straightforward computation gives

$$K \circ \partial + \partial \circ K = \text{id} : \Omega^m(\widehat{\mathcal{G}}^{(i)}) \rightarrow \Omega^m(\widehat{\mathcal{G}}^{(i)}), \text{ for } i \geq 1.$$

□

We shall be interested in the following differentiable situation of a \mathcal{G} -action on a smooth manifold P . A \mathcal{G} equivariant submersion is given by a submersion $\pi : P \rightarrow \mathcal{G}^{(0)}$ and a diffeomorphism $\lambda_\gamma : P_{r(\gamma)} \rightarrow P_{s(\gamma)}$ for every $\gamma \in \mathcal{G}$ such that $\lambda_{\gamma_1 \gamma_2} = \lambda_{\gamma_2} \circ \lambda_{\gamma_1}$ and $\lambda_{\gamma^{-1}} = \lambda_\gamma^{-1}$. We set $\lambda_\gamma(p) := p\gamma$ for $p \in P_{r(\gamma)}$. Given such an action we can form a new étale groupoid $\mathcal{P} = P \rtimes \mathcal{G}$ with the space of objects P and the morphisms given by pairs $(p, \gamma) \in P \times \mathcal{G}$ such that $\pi(p) = r(\gamma)$ with $r(p, \gamma) = p$, $s(p, \gamma) = p\gamma$. The map π defines a morphism of groupoids $(p, \gamma) \mapsto \gamma$. The groupoid \mathcal{P} is proper if and only if the \mathcal{G} -action on P is proper.

The associated simplicial space is described explicitly as follows:

$$\mathcal{P}^{(k)} = (P \rtimes \mathcal{G})^{(k)} = \{(p, \gamma_1, \dots, \gamma_k) \in P \times \mathcal{G}^{(k)} \mid \pi(p) = r(\gamma_1)\}.$$

The face maps $\partial_i : \mathcal{P}^{(k)} \rightarrow \mathcal{P}^{(k-1)}$ are given explicitly by

$$\partial_i(p, \gamma_1, \dots, \gamma_k) = \begin{cases} (p\gamma_1, \gamma_2, \dots, \gamma_k) & \text{for } i = 0 \\ (p, \gamma_1, \dots, \gamma_i \gamma_{i+1}, \dots, \gamma_k) & \text{for } 0 < i < k \\ (p, \gamma_1, \dots, \gamma_{k-1}) & \text{for } i = k \end{cases}$$

π induces submersions $\pi^{(n)} : \mathcal{P}^{(n)} \rightarrow \mathcal{G}^{(n)}$ by

$$\pi^{(n)}(p, \gamma_1, \dots, \gamma_k) = (\gamma_1, \dots, \gamma_k)$$

These submersions are compatible with the face maps.

1.2. The Dupont-Sullivan bicomplex. We shall also use the Dupont-Sullivan bicomplex associated with \mathcal{G} . Denote by Δ^k the standard k -simplex

$$\Delta^k := \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} \mid 0 \leq t_i, \sum_{i=0}^k t_i = 1\}.$$

and by $\delta_i : \Delta^{k-1} \rightarrow \Delta^k$ the face maps

$$\delta_i(t_0, \dots, t_{k-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1})$$

The algebra of Dupont-Sullivan forms was introduced in [31] and will be denoted here by $\Omega_{DS}^*(\mathcal{G})$. So $\Omega_{DS}^*(\mathcal{G}) \subset \prod_k \Omega^*(\mathcal{G}^{(k)}) \otimes \Omega^*(\Delta^k)$ and an element of $\Omega_{DS}^*(\mathcal{G})$ is given by a collection $\omega = (\omega_k)_k$ of forms

$$\omega_k \in \Omega^*(\mathcal{G}^{(k)}) \otimes \Omega^*(\Delta^k) = \Omega^*(\mathcal{G}^{(k)} \times \Delta^k)$$

satisfying for every $k \geq 1$ the following relation

$$(\text{id} \times \delta_i)^* \omega_k = (\partial_i \times \text{id})^* \omega_{k-1} \in \Omega^*(\mathcal{G}^{(k)} \times \Delta^{k-1}).$$

There is a natural bigrading on $\Omega_{DS}^*(\mathcal{G})$ given by

$$\Omega_{DS}^{i,j}(\mathcal{G}) = \Omega_{DS}^*(\mathcal{G}) \cap \left(\prod_k \Omega^i(\mathcal{G}^{(k)}) \otimes \Omega^j(\Delta^k) \right),$$

and we set as usual $\Omega_{DS}^m(\mathcal{G}) := \bigoplus_{i+j=m} \Omega_{DS}^{i,j}(\mathcal{G})$. The differential is given by $d_{\mathcal{G}} + d_{\Delta}$ where $d_{\mathcal{G}}$ is de Rham differential on \mathcal{G} and d_{Δ} is de Rham differential on Δ . We will also consider the complex $\Omega_{\bullet}^{\Delta}(\mathcal{G}) = \Omega_{DS}^*(\mathcal{G})[u]$ where the differential has now degree -1 and is given by $u(d_{\mathcal{G}} + d_{\Delta})$.

The dual version of currents will also be needed. We refer the reader to A.0.1 for the definition of the spaces $\mathcal{C}^{\bullet}(B) := \mathcal{C}_*(B)[u^*]$ and $\mathcal{C}_*(B)[(u^*)^{-1}, u^*] := \mathcal{C}^{\bullet}(B)[(u^*)^{-1}]$, where u^* is a formal variable of degree $+2$. For an étale map $f: B \rightarrow B'$ there is a well-defined pull-back

$$f^*: \mathcal{C}^{\bullet}(B') \rightarrow \mathcal{C}^{\bullet}(B),$$

dual to the “integration” along the fibers. We can therefore define the space of simplicial smooth currents $\mathcal{C}_{\Delta}^{\bullet}(\mathcal{G}) \subset \prod_k \mathcal{C}_*(\mathcal{G}^{(k)})[(u^*)^{-1}, u^*] \otimes \Omega^*(\Delta^k)$ as given by a collection $\{\xi_k\}$ where

$$\xi_k \in \mathcal{C}_*(\mathcal{G}^{(k)})[(u^*)^{-1}, u^*] \otimes \Omega^*(\Delta^k)$$

satisfying for every k

$$(\text{id} \otimes \delta_i^*) \xi_k = (\partial_i^* \otimes \text{id}) \xi_{k-1} \in \mathcal{C}_*(\mathcal{G}^{(k)})[(u^*)^{-1}, u^*] \otimes \Omega(\Delta^{k-1}).$$

The differential is given by $d^t = u^* d_{\mathcal{G}}^t + d_{\Delta}$.

Set for $\alpha \otimes \beta \in \Omega^i(\mathcal{G}^{(k)})[u] \otimes \Omega^j(\Delta^k)$, $\xi \otimes \omega \in \mathcal{C}_*(\mathcal{G}^{(k)})[u^*] \otimes \Omega^*(\Delta^k)$ (cf. A.0.1).

$$\langle (\xi \otimes \omega), (\alpha \otimes \beta) \rangle = (-1)^{\deg \alpha \deg \omega} \langle (u^*)^{-j} \xi, \alpha \rangle \int_{\Delta^k} \omega \wedge \beta,$$

We then have, with $\phi \in \Omega^*(\mathcal{G}^{(k)})[u] \otimes \Omega^*(\Delta^k)$, $c \in \mathcal{C}_*(\mathcal{G}^{(k)})[u^*] \otimes \Omega^*(\Delta^k)$,

$$\langle d_{\mathcal{G}}^t c, \phi \rangle + (-1)^{\deg c} \langle c, d_{\mathcal{G}} \phi \rangle = 0$$

and

$$\langle d_{\Delta} c, \phi \rangle + (-1)^{\deg c} \langle c, d_{\Delta} \phi \rangle = \sum_{i=0}^k (-1)^i \langle (\text{id} \otimes \delta_i^*) c, (\text{id} \otimes \delta_i^*) \phi \rangle$$

Define also a cap product $\mathcal{C}_{\Delta}^{\bullet}(\mathcal{G}) \otimes \Omega_{\bullet}^{\Delta}(\mathcal{G}) \rightarrow \mathcal{C}_{\Delta}^{\bullet}(\mathcal{G})$ by setting

$$(\xi \otimes \omega) \cap (\alpha \otimes \beta) = (-1)^{\deg \alpha \deg \omega} (u^*)^{-j} (\xi \cap \alpha) \otimes (\omega \wedge \beta)$$

If $\Omega \in \Omega_{DS}^3(\mathcal{G})$ is a closed form we define the associated twisted complexes which compute the twisted cohomology/homology of the classifying space $B\mathcal{G}$. So, $(\Omega_{\bullet}^{\Delta}(\mathcal{G}), d_{\Omega})$ is obtained by considering the differential $d_{\Omega} = u(d_{\mathcal{G}} + d_{\Delta}) + u^2 \Omega \wedge \cdot$, where the wedge is induced from the wedge products on the manifolds $\mathcal{G}^{(k)} \times \Delta^k$. Similarly, we define the twisted complex $(\mathcal{C}_{\Delta}^{\bullet}(\mathcal{G}), d_{\Omega}^t)$ by setting for $c \in \mathcal{C}_{\Delta}^{\bullet}(\mathcal{G})$

$$(1.2.1) \quad d_{\Omega}^t c = d^t c + (-1)^{\deg c} c \cap \Omega = u^* d_{\mathcal{G}}^t c + d_{\Delta} c + (-1)^{\deg c} c \cap \Omega.$$

The homology of this complex will be denoted $H_{\Omega}^{\bullet}(B\mathcal{G})$. We point out that only 3 forms Ω with no $(3, 0)$ component arise in the present paper.

2. GERBES AND TWISTED BUNDLES

We review in this section some background material about connections on gerbes and their morphisms. We also recall many results that will be used in the next sections. Our companion paper is [4] as we shall use many constructions from there. We give here only the general overview, referring the reader to [14], [39], [49] and [50] for the details. The differential geometry of not necessarily abelian gerbes is described in [11].

2.1. Gerbes and connections.

We shall describe the gerbes in terms of their descent data.

Let M be a smooth manifold. Given an open cover $(U_\alpha)_{\alpha \in \Lambda}$ of M , we set as usual

$$U_{\alpha_1 \dots \alpha_k} = \bigcap_{1 \leq j \leq k} U_{\alpha_j}.$$

Definition 2.1. A descent datum for a gerbe \mathcal{L} on M is the collection $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ where $(U_\alpha)_{\alpha \in \Lambda}$ is an open cover of M , $(\mathcal{L}_{\alpha\beta} \rightarrow U_{\alpha\beta})_{\alpha, \beta \in \Lambda}$ is a collection of line bundles and $\mu_{\alpha\beta\gamma} : \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \rightarrow \mathcal{L}_{\alpha\gamma}$ is a collection of line bundle isomorphisms over each triple intersection $U_{\alpha\beta\gamma}$ such that over each quadruple intersection $U_{\alpha\beta\gamma\delta}$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\delta} & \xrightarrow{\mu_{\alpha\beta\gamma} \otimes \text{id}} & \mathcal{L}_{\alpha\gamma} \otimes \mathcal{L}_{\gamma\delta} \\ \text{id} \otimes \mu_{\beta\gamma\delta} \downarrow & & \downarrow \mu_{\alpha\gamma\delta} \\ \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\delta} & \xrightarrow{\mu_{\alpha\beta\delta}} & \mathcal{L}_{\alpha\delta} \end{array}$$

Notice that we don't need to assume in this definition that the open sets $(U_\alpha)_{\alpha \in \Lambda}$ are contractible, however, our covers will always be good covers [10].

Given two descent data $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$ on the same open cover $\{U_\alpha\}$, an isomorphism between them is given by line bundles S_α on U_α and isomorphisms of line bundles $\lambda_{\alpha\beta} : S_\alpha^{-1} \otimes \mathcal{L}_{\alpha\beta} \otimes S_\beta \rightarrow \mathcal{L}'_{\alpha\beta}$ over $U_{\alpha\beta}$ so that the following diagram commutes

$$\begin{array}{ccc} S_\alpha^{-1} \otimes \mathcal{L}_{\alpha\beta} \otimes S_\beta \otimes S_\beta^{-1} \otimes \mathcal{L}_{\beta\gamma} \otimes S_\gamma & \xrightarrow{\text{id} \otimes \mu_{\alpha\beta\gamma} \otimes \text{id}} & S_\alpha^{-1} \otimes \mathcal{L}_{\alpha\gamma} \otimes S_\gamma \\ \lambda_{\alpha\beta} \otimes \lambda_{\beta\gamma} \downarrow & & \downarrow \lambda_{\alpha\gamma} \\ \mathcal{L}'_{\alpha\beta} \otimes \mathcal{L}'_{\beta\gamma} & \xrightarrow{\mu'_{\alpha\beta\gamma}} & \mathcal{L}'_{\alpha\gamma} \end{array}$$

It is clear how to compose such (iso)morphisms.

Definition 2.2. Given two isomorphisms $(S_\alpha, \lambda_{\alpha\beta})$ and $(S'_\alpha, \lambda'_{\alpha\beta})$ between $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$, a 2-morphism between them is a collection of line bundle isomorphisms $\nu_\alpha : S_\alpha \rightarrow S'_\alpha$ such that

$$(2.1.1) \quad \lambda'_{\alpha\beta} \circ (\nu_\alpha^{-1} \otimes \text{id} \otimes \nu_\beta) = \lambda_{\alpha\beta}$$

where we denote by ν_α^{-1} the isomorphism $S_\alpha^{-1} \rightarrow (S'_\alpha)^{-1}$ induced by ν_α .

Notice that Equation 2.1.1 allows to define the collection $(\lambda_{\alpha\beta})$ which then satisfies the commutativity of the previous diagram. More generally, a 2-morphism can be defined between any isomorphisms of gerbes as follows.

Definition 2.3. Consider an isomorphism $\varsigma_1 = (S_{1,\alpha}, \lambda_{1,\alpha\beta})$ between the descent data $(U_\alpha, \mathcal{L}_{1,\alpha\beta}, \mu_{1,\alpha\beta\gamma})$ and $(U_\alpha, \mathcal{L}'_{1,\alpha\beta}, \mu'_{1,\alpha\beta\gamma})$ and, an isomorphism $\varsigma_2 = (S'_{2,\alpha}, \lambda'_{2,\alpha\beta})$ between the descent data $(U_\alpha, \mathcal{L}_{2,\alpha\beta}, \mu_{2,\alpha\beta\gamma})$ and $(U_\alpha, \mathcal{L}'_{2,\alpha\beta}, \mu'_{2,\alpha\beta\gamma})$. Then, a 2-morphism between them is given by the following data:

- An isomorphism $\varsigma = (S_\alpha, \lambda_{\alpha\beta})$ between the descent data $(U_\alpha, \mathcal{L}_{1,\alpha\beta}, \mu_{1,\alpha\beta\gamma})$ and $(U_\alpha, \mathcal{L}_{2,\alpha\beta}, \mu_{2,\alpha\beta\gamma})$ and an isomorphism $\varsigma' = (S'_\alpha, \lambda'_{\alpha\beta})$ between the descent data $(U_\alpha, \mathcal{L}'_{1,\alpha\beta}, \mu'_{1,\alpha\beta\gamma})$ and $(U_\alpha, \mathcal{L}'_{2,\alpha\beta}, \mu'_{2,\alpha\beta\gamma})$, in the sense of Definition 2.1 above;
- A 2-morphism ν between the corresponding descent data for the composite isomorphism $\varsigma' \circ \varsigma_1$ and for the composite isomorphism $\varsigma_2 \circ \varsigma$, in the sense of Definition 2.2.

We shall refer to such 2-morphism by $(\nu, \varsigma, \varsigma')$. Definition 2.2 corresponds to the identity gerbes isomorphisms and in this case we refer to the 2-morphism simply by ν .

Let $(V_i, \varrho)_{i \in I}$ be a refinement of the open cover $(U_\alpha)_{\alpha \in \Lambda}$ of M . So $(V_i)_{i \in I}$ is an open cover of M with $\varrho : I \rightarrow \Lambda$ such that $V_i \subset U_{\varrho(i)}$. Then restriction to the refinement $(V_i, \varrho)_{i \in I}$ of $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ is the descent datum $\mathcal{L}' = (V_i, \mathcal{L}'_{ij}, \mu'_{ijk})$ given by:

$$\mathcal{L}'_{ij} := \mathcal{L}_{\varrho(i)\varrho(j)}|_{V_{ij}} \text{ and } \mu'_{ijk} := \mu_{\varrho(i)\varrho(j)\varrho(k)}|_{V_{ijk}}.$$

Similarly one defines restriction of the isomorphisms and 2-morphisms to a refinement. We do not distinguish between a descent datum and its restriction to a refinement. Thus for instance, the isomorphism between two descent data $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$ is an isomorphism between their restrictions to some common refinement of $\{U_\alpha\}$ and $\{U'_\alpha\}$. Equivalence classes of descent data are thus well defined, and an equivalence class of (Dixmier-Douady) gerbes on M is an equivalence class of such descent data on M .

Recall more, a gerbe can be defined as a maximal collection of descent data \mathcal{D}_i , $i \in A$ together with the isomorphisms $s_{ij} : \mathcal{D}_j \rightarrow \mathcal{D}_i$ for each $i, j \in A$ and 2-morphisms $\nu_{ijk} : s_{ij}s_{jk} \rightarrow s_{ik}$ satisfying the natural associativity condition. We refer the reader to the book [14] for the details.

If the cover $\{U_\alpha\}$ is good [10], all the bundles $\mathcal{L}_{\alpha\beta}$ are trivializable. After choice of such a trivialization, the collection $(\mu_{\alpha\beta\gamma})$ can be viewed as a Čech 2-cochain with coefficients in the sheaf $\underline{\mathbb{C}}^*$ of smooth functions with values in the nonzero complex numbers \mathbb{C}^* . We can and will, unless otherwise specified, always work with good covers. The compatibility condition over $U_{\alpha\beta\gamma\delta}$ tells us that μ is a 2-cocycle and hence defines a cohomology class $[\mu] \in H^2(M; \underline{\mathbb{C}}^*) \cong H^3(M, \mathbb{Z})$. This class is a well defined invariant of the gerbe called the Dixmier-Douady class. We denote this class by $[\mathcal{L}]$. Every class in $H^3(M, \mathbb{Z})$ is a class of a gerbe defined by this class uniquely up to an isomorphism (see again [14]).

Given a smooth map $f : M' \rightarrow M$ between smooth manifolds M' and M , we can pull-back any descent datum for a gerbe on M to a descent datum on M' . The pull backs of isomorphic descent data are isomorphic and thus we obtain a well-defined pull-back of a gerbe. Clearly the Dixmier-Douady class of the pull-back is the pull-back of the Dixmier-Douady class.

An unitary descent datum is $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ together with a choice of metric on each $\mathcal{L}_{\alpha\beta}$ such that each $\mu_{\alpha\beta\gamma}$ is an isometry. A notion of unitary equivalence of two unitary descent data on the same open cover U_α is obtained from the notion of equivalence above by requiring that each line bundle S_α is Hermitian and each $\lambda_{\alpha\beta}$ is an isometry. The definition of 2-morphisms is modified by requiring each ν_α to be an isometry. It is clear that the restriction of a unitary descent datum to a refinement is again unitary. Then a unitary gerbe is an equivalence class of unitary descent data in the sense described above.

Lemma 2.4. *Let $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ be a descent datum on M . There exists a collection $(\nabla_{\alpha\beta})$ of connections on $(\mathcal{L}_{\alpha\beta})$ such that for any $(\alpha, \beta, \gamma) \in \Lambda^3$ with $U_{\alpha\beta\gamma} \neq \emptyset$:*

$$\mu_{\alpha\beta\gamma}^* \nabla_{\alpha\gamma} = \nabla_{\alpha\beta} \otimes \text{id} + \text{id} \otimes \nabla_{\beta\gamma}.$$

If the descent datum is unitary each $\nabla_{\alpha\beta}$ can be chosen Hermitian.

Lemma 2.5. *Let $(\nabla_{\alpha\beta})$ be as above, and denote by $\omega_{\alpha\beta} = \nabla_{\alpha\beta}^2$ the curvatures of the connections $\nabla_{\alpha\beta}$. Then there exists a collection of differential 2-forms $\omega_\alpha \in \Omega^2(U_\alpha)$ such that*

$$\omega_{\alpha\beta} = \omega_\alpha - \omega_\beta, \quad \text{for } U_{\alpha\beta} \neq \emptyset.$$

Definition 2.6. The collection $\nabla^\mathcal{L} = (\nabla_{\alpha\beta})$ is a connective structure on \mathcal{L} , while $\omega^\mathcal{L} = (\omega_\alpha)$ is a curving (compatible with $\nabla^\mathcal{L}$). We say that the collection $(\nabla_{\alpha\beta}, \omega_\alpha)$ is a connection on the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$.

If $\xi \in \Omega^2(M)$ then $\omega^\mathcal{L} + \xi := (\omega_\alpha + \xi)$ is another curving compatible with $\nabla^\mathcal{L}$. A connection on a descent datum yields a connection on its restriction to a refinement in an obvious manner. A connection will be identified with its restriction.

Definition 2.7. Let $\varsigma = (S_\alpha, \lambda_{\alpha\beta})$ be an isomorphism of descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ with connective structure $\{\nabla_{\alpha\beta}\}$ and descent datum $(U_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$ with connective structure $\{\nabla'_{\alpha\beta}\}$. A connection on ς is a collection of connections (∇_α) on (S_α) , satisfying the equality

$$(2.1.2) \quad (\lambda_{\alpha\beta})^* \nabla'_{\alpha\beta} = \nabla_\alpha^* \otimes \text{id} + \text{id} \otimes \nabla_{\alpha\beta} + \text{id} \otimes \text{id} \otimes \nabla_\beta.$$

Here ∇_α^* denotes the dual connection on S_α^{-1} .

If $\{\nabla_\alpha\}$ is a connection on ς and $\xi \in \Omega^1(M)$ then $\{\nabla_\alpha + \xi\}$ is another such connection.

Let $\varsigma = (S_\alpha, \lambda_{\alpha\beta}, \nabla_\alpha)$ be a morphism of descent data with a connection, and let $\varpi_\alpha = \nabla_\alpha^2$ be the associated collection of curvatures.

Lemma 2.8. *There exists a global form $c(\varsigma) \in \Omega^2(M)$ such that*

$$c(\varsigma)|_{U_\alpha} = \omega'_\alpha + \varpi_\alpha - \omega_\alpha.$$

We say that ς is a flat morphism if $c(\varsigma) = 0$, i.e. if

$$(2.1.3) \quad \omega_\alpha = \omega'_\alpha + \varpi_\alpha.$$

Lemma 2.9. *Let ς be an isomorphism between the two gerbes \mathcal{L} and \mathcal{L}' with connective structures. Then there exists a connection on ς .*

Proof. We can assume that the gerbes and the isomorphism are represented by data over the same good open cover $(U_\alpha)_\alpha$. So $(U_\alpha, S_\alpha, \lambda_{\alpha\beta})$ is an isomorphism between $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$. We are given connective structures $(\nabla_{\alpha\beta})$ and $(\nabla'_{\alpha\beta})$ on these descent data for \mathcal{L} and \mathcal{L}' respectively, and we fix a collection of connections $(\nabla_\alpha)_\alpha$ on the line bundles $(S_\alpha)_\alpha$. Denote by $\bar{\nabla}_{\alpha\beta}$ the connection defined out of the collections $(\nabla'_{\alpha\beta})$ and (∇_α) by Equation 2.1.2:

$$\bar{\nabla}'_{\alpha\beta} := (\lambda_{\alpha\beta}^{-1})^* (\nabla_\alpha^* \otimes \text{id} \otimes \text{id} + \text{id} \otimes \nabla_{\alpha\beta} \otimes \text{id} + \text{id} \otimes \text{id} \otimes \nabla_\beta).$$

Then the difference $\nabla_{\alpha\beta} - \bar{\nabla}_{\alpha\beta}$ is a Čech 2-cocycle with values in the soft sheaf of 1-forms, hence there exists a collection $(\varphi_\alpha)_\alpha$ of 1-forms on $(U_\alpha)_\alpha$ such that over $U_{\alpha\beta} \neq \emptyset$, we have

$$\nabla_{\alpha\beta} - \bar{\nabla}_{\alpha\beta} = \varphi_\alpha - \varphi_\beta.$$

It is then easy to check that the collection of connections $(\nabla_\alpha + \varphi_\alpha)_\alpha$ allows to endow ς with the structure of an isomorphism of our gerbes with connective structures. \square

If $(S_\alpha, \lambda_{\alpha\beta}, \nabla_\alpha)$ and $(S'_\alpha, \lambda'_{\alpha\beta}, \nabla'_\alpha)$ are two morphisms between the descent data with connections as above, the 2-morphisms between them are thus the same as the 2-morphisms between $(S_\alpha, \lambda_{\alpha\beta})$ and $(S'_\alpha, \lambda'_{\alpha\beta})$.

Assume that we are given two gerbes \mathcal{L} and \mathcal{L}' and two morphisms ς, ς' between them. On an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ represent ς by $(S_\alpha, \lambda_{\alpha\beta}, \nabla_\alpha)$ and ς' by $(S'_\alpha, \lambda'_{\alpha\beta}, \nabla'_\alpha)$. For a 2-morphism $\nu: \varsigma \rightarrow \varsigma'$, we define the 1-form $c(\nu)|_{U_\alpha}$ by

$$c(\nu)|_{U_\alpha} = \nu_\alpha^* \nabla'_\alpha - \nabla_\alpha$$

Lemma 2.10. *The collection of local 1-forms $(c(\nu)|_{U_\alpha})_\alpha$ defines a global form $c(\nu) \in \Omega^1(M)$.*

Proof. Indeed, notice that the isomorphism

$$\nu_\alpha \otimes \text{id} \otimes \nu_\beta^{-1} : S_\alpha \otimes \mathcal{L}'_{\alpha\beta} \otimes S_\beta^{-1} \longrightarrow S'_\alpha \otimes \mathcal{L}'_{\alpha\beta} \otimes S'^{-1}_\beta$$

pulls back the connection $\nabla'_\alpha \otimes \text{id} + \text{id} \otimes \nabla'_{\alpha\beta} \otimes \text{id} + \text{id} \otimes \nabla'^{-1}_\beta$ to the connection $\nabla_\alpha \otimes \text{id} + \text{id} \otimes \nabla'_{\alpha\beta} \otimes \text{id} + \text{id} \otimes \nabla_\beta^{-1}$. From this the following relation holds

$$(\nu_\alpha^* \nabla'_\alpha - \nabla_\alpha) \otimes \text{id} + \text{id} \otimes ((\nu_\beta^{-1})^* \nabla'^{-1}_\beta - \nabla_\beta^{-1}) = 0,$$

and this ends the proof. \square

We now state the following lemma which is proved in [4][page 7]:

Lemma 2.11. *Let \mathcal{L} be a gerbe represented by a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$. Choose a connection on \mathcal{L} represented by the collection of connections $(\nabla_{\alpha\beta}, \omega_\alpha)$ on the descent datum.*

- (1) *There exists a well-defined closed form $\Omega \in \Omega^3(M)$ – curvature 3-form of the connection – such that $\Omega|_{U_\alpha} = \frac{d\omega_\alpha}{2\pi i}$.*
- (2) *Let $(\nabla'_{\alpha\beta}, \omega'_\alpha)$ be another connection on \mathcal{L} and let Ω' be the corresponding curvature 3-form. Then there exists a canonical $\eta \in \Omega^2(M)/d\Omega^1(M)$ such that $\Omega' = \Omega + d\eta$.*

- (3) Let $(\nabla_{\alpha\beta}, \omega_\alpha)$, $(\nabla'_{\alpha\beta}, \omega'_\alpha)$ and $(\nabla''_{\alpha\beta}, \omega''_\alpha)$ be 3 connections on \mathcal{L} with the corresponding curvature 3-forms Ω , Ω' and Ω'' . Let $\eta, \eta', \eta'' \in \Omega^2(M)/d\Omega^1(M)$ be the canonical elements constructed above such that $\Omega' - \Omega = d\eta$, $\Omega'' - \Omega' = d\eta'$, $\Omega'' - \Omega = d\eta''$. Then $\eta'' = \eta + \eta'$.

Choosing a unitary descent datum for \mathcal{L} and a hermitian connection $(\nabla_{\alpha\beta}, \omega_\alpha)$, a classical argument shows that the 3-form Ω is a de Rham representative of the Dixmier-Douady class of the gerbe. An argument similar to the proof of Lemma 2.4 shows the following proposition which gathers the main properties that will be used in the sequel. The proofs are straightforward consequences of the definitions and are omitted:

Proposition 2.12.

- (1) Let $\varsigma: \mathcal{L} \rightarrow \mathcal{L}'$ be a morphism of gerbes with connections. Then

$$dc(\varsigma) = \Omega(\mathcal{L}') - \Omega(\mathcal{L}).$$

- (2) Let $\varsigma: \mathcal{L} \rightarrow \mathcal{L}'$, $\varsigma': \mathcal{L}' \rightarrow \mathcal{L}''$ be morphisms of gerbes with connections. Then

$$c(\varsigma' \circ \varsigma) = c(\varsigma') + c(\varsigma).$$

- (3) Let $\varsigma: \mathcal{L} \rightarrow \mathcal{L}'$, $\varsigma': \mathcal{L} \rightarrow \mathcal{L}'$ be morphisms of gerbes with connections, and $\nu: \varsigma \rightarrow \varsigma'$ a 2-morphism. Then

$$dc(\nu) = c(\varsigma) - c(\varsigma').$$

- (4) The correspondence $\nu \mapsto c(\nu)$ is additive under the horizontal and vertical compositions of 2-morphisms.

2.2. Twisted and transversally twisted bundles.

Definition 2.13. A descent datum for a twisted vector bundle \mathcal{E} consists of a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ for a gerbe \mathcal{L} together with a collection $(\mathcal{E}_\alpha \rightarrow U_\alpha)_{\alpha \in \Lambda}$ of vector bundles and a collection of vector bundle isomorphisms $\varphi_{\alpha\beta}: \mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta} \xrightarrow{\cong} \mathcal{E}_\beta$ such that for every α, β, γ the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} & \xrightarrow{\text{id} \otimes \mu_{\alpha\beta\gamma}} & \mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\gamma} \\ \varphi_{\alpha\beta} \otimes \text{id} \downarrow & & \downarrow \varphi_{\alpha\gamma} \\ \mathcal{E}_\beta \otimes \mathcal{L}_{\beta\gamma} & \xrightarrow{\varphi_{\beta\gamma}} & \mathcal{E}_\gamma \end{array}$$

Restriction of the descent datum for \mathcal{E} to a refinement is given by the restriction of the descent datum for \mathcal{L} together with restriction of the vector bundles \mathcal{E}_α and the isomorphisms $\varphi_{\alpha\beta}$.

Definition 2.14. An (iso)morphism between two descent data $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ and $(U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma}, \mathcal{E}'_\alpha, \varphi'_{\alpha\beta})$ is given by the collection $(\rho_\alpha, S_\alpha, \lambda_{\alpha\beta})$ where $(S_\alpha, \lambda_{\alpha\beta})$ is an isomorphism between $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$ and $\rho_\alpha: \mathcal{E}_\alpha \otimes S_\alpha \rightarrow \mathcal{E}'_\alpha$ is a collection of

(iso)morphisms such that the following diagram commutes

$$\begin{array}{ccc}
 \mathcal{E}_\alpha \otimes S_\alpha \otimes S_\alpha^{-1} \otimes \mathcal{L}_{\alpha\beta} \otimes S_\beta & \xrightarrow{\varphi_{\alpha\beta} \otimes \text{id}} & \mathcal{E}_\beta \otimes S_\beta \\
 \rho_\alpha \otimes \lambda_{\alpha\beta} \downarrow & & \downarrow \rho_\beta \\
 \mathcal{E}'_\alpha \otimes \mathcal{L}'_{\alpha\beta} & \xrightarrow{\varphi'_{\alpha\beta\gamma}} & \mathcal{E}'_\beta
 \end{array}$$

An isomorphism between two descent data on two different covers is defined as an isomorphism between their restrictions on a common refinement.

Definition 2.15. A 2-morphism between two isomorphisms $(\rho_\alpha, S_\alpha, \lambda_{\alpha\beta})$ and $(\rho'_\alpha, S'_\alpha, \lambda'_{\alpha\beta})$ is the 2-morphism between the corresponding isomorphisms of the gerbe descent data, i.e. collection of morphisms $\nu_\alpha: S_\alpha \rightarrow S'_\alpha$ subject to (2.1.1), such that we have in addition

$$\rho'_\alpha \circ (\text{id} \otimes \nu_\alpha) = \rho_\alpha.$$

A twisted bundle is then defined as an equivalence class of descent data of twisted vector bundles. “Forgetting” the bundle data we obtain from the descent datum for a twisted vector bundle a descent datum for a gerbe, and the same applies to morphisms and 2-morphisms. We say that \mathcal{E} is an \mathcal{L} -twisted vector bundle if “forgetting” the bundle data one obtains the equivalence class of the gerbe descent data defining \mathcal{L} .

Assume now that the gerbe \mathcal{L} is unitary. An Hermitian descent datum for \mathcal{E} consists of a unitary descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ for \mathcal{L} and a collection h_α of metrics on \mathcal{E}_α such that the maps $\varphi_{\alpha\beta}$ are isometries. One obtains a notion of isomorphism of Hermitian descent data by requiring the ρ_α ’s to be isometries. An Hermitian twisted bundle is then an equivalence class of Hermitian descent data.

Remark 2.16. *Given a gerbe $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ on M , it is well known that (finite dimensional) twisted vector bundles exist if and only if the gerbe is torsion (see e.g. [30, 40] and references therein).*

The following lemma from [4] will be used later.

Lemma 2.17. *Let \mathcal{L} be a gerbe on M and \mathcal{E} an \mathcal{L} -twisted bundle. Let $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ be any descent datum for \mathcal{L} . Then there exists a descent datum for \mathcal{E} isomorphic to the one of the form $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$.*

We now fix a gerbe \mathcal{L} with a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ on the smooth manifold M together with a twisted vector bundle \mathcal{E} represented by $(\mathcal{E}_\alpha, \varphi_{\alpha\beta})$. We denote by \mathcal{A}_α the collection of bundles of algebras

$$\mathcal{A}_\alpha := \text{End}(\mathcal{E}_\alpha), \quad \alpha \in \Lambda.$$

For any $U_{\alpha\beta} \neq \emptyset$, we have a canonical vector bundle isomorphism over $U_{\alpha\beta}$

$$\rho_{\alpha\beta} : \text{End}(\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta}) \longrightarrow \mathcal{A}_\alpha,$$

extending the canonical isomorphism $\text{End}(\mathcal{L}_{\alpha\beta}) \simeq U_{\alpha\beta} \times \mathbb{C}$. Therefore, the bundle isomorphism $\varphi_{\alpha\beta}$ together with the identification $\rho_{\alpha\beta}$ induce the isomorphism of algebra bundles

over $U_{\alpha\beta}$ given by

$$\mathcal{A}_\beta \xrightarrow{\varphi_{\alpha\beta}^*} \text{End}(\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta}) \xrightarrow{\rho_{\alpha\beta}} \mathcal{A}_\alpha.$$

We denote this isomorphism simply by $\varphi_{\alpha\beta}^*$. It is then easy to check that

$$\varphi_{\alpha\beta}^* \circ \varphi_{\beta\gamma}^* = \varphi_{\alpha\gamma}^*, \quad \text{over } U_{\alpha\beta\gamma}.$$

Therefore, the collection $\{\mathcal{A}_\alpha\}$ defines a bundle \mathcal{A} of algebras over M , which is denoted $\text{End}(\mathcal{E})$. It is an easy exercise to check that the isomorphism class of $\text{End}(\mathcal{E})$ depends only on the isomorphism class of the \mathcal{L} -twisted vector bundle \mathcal{E} .

Definition 2.18. The bundle $\mathcal{A} = \text{End}(\mathcal{E})$ is called the Azumaya bundle associated with the \mathcal{L} -twisted vector bundle \mathcal{E} .

If \mathcal{E} is a Hermitian twisted bundle, each of the bundles $\text{End}(\mathcal{E}_\alpha)$ is a bundle of $*$ -algebras, with the $*$ -operation given by taking the adjoint endomorphism. This induces a structure of a bundle of $*$ -algebras on $\text{End}(\mathcal{E})$.

Note that for every α we have the trace $\text{tr}_\alpha: \text{End}(\mathcal{E}_\alpha) \rightarrow C^\infty(U_\alpha)$. For $a \in \Gamma(U_{\alpha\beta}; \mathcal{A}_\beta)$ $\text{tr}_\alpha(\varphi_{\alpha\beta}^*(a)) = \text{tr}_\beta(a)$. Therefore we obtain the trace $\text{tr}: \text{End}(\mathcal{E}) \rightarrow C^\infty(M)$ defined by

$$\text{tr}(a)|_{U_\alpha} = \text{tr}_\alpha(a|_{U_\alpha}) \text{ for } a \in \text{End}(\mathcal{E})$$

If the bundle \mathcal{E} is \mathbb{Z}_2 -graded then $\text{End}(\mathcal{E})$ is a bundle of \mathbb{Z}_2 -graded algebras with a supertrace

$$\text{str}: \text{End}(\mathcal{E}) \longrightarrow C^\infty(M).$$

Notice that, more generally, if \mathcal{E} and \mathcal{E}' are \mathcal{L} -twisted bundles, we have a well defined vector bundle $\text{Hom}(\mathcal{E}, \mathcal{E}')$, defined similarly by $\text{Hom}(\mathcal{E}, \mathcal{E}')|_{U_\alpha} = \text{Hom}(\mathcal{E}|_{U_\alpha}, \mathcal{E}'|_{U_\alpha})$.

Definition 2.19. Let as before $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ be a descent datum for the \mathcal{L} -twisted vector bundle \mathcal{E} on M . A connection on $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ is a collection $(\nabla_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ where $(\nabla_{\alpha\beta}, \omega_\alpha)$ is a connection on $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and each ∇_α is a connection on \mathcal{E}_α such that the identities

$$(2.2.1) \quad \varphi_{\alpha\beta}^* \nabla_\beta = \nabla_\alpha \otimes \text{id} + \text{id} \otimes \nabla_{\alpha\beta}$$

hold whenever $U_{\alpha\beta} \neq \emptyset$.

Again the following lemma is proved in [4][page 11]:

Lemma 2.20. *Let $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ be a descent datum for an \mathcal{L} -twisted vector bundle \mathcal{E} . Then every connection $(\nabla_{\alpha\beta}, \omega_\alpha)$ on the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ for \mathcal{L} can be extended to a connection for the descent datum for \mathcal{E} .*

Definition 2.21. An isomorphism between two descent data $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ and $(U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma}, \mathcal{E}'_\alpha, \varphi'_{\alpha\beta})$ for \mathcal{E} , with connections $(\nabla_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ and $(\nabla'_\alpha, \nabla'_{\alpha\beta}, \omega'_\alpha)$, is given by the collections $S = (\rho_\alpha, S_\alpha, \nabla_\alpha^S, \lambda_{\alpha\beta})$ where $(\rho_\alpha, S_\alpha, \lambda_{\alpha\beta})$ is an isomorphism between the descent data without connections, ∇_α^S are connections on S_α such that $\varsigma = (S_\alpha, \lambda_{\alpha\beta}, \nabla_\alpha^S)$ is a morphism of the corresponding gerbe descent data with connective structures (i.e. the equation (2.1.2) is satisfied) and the equality

$$(2.2.2) \quad \rho_\alpha^* \nabla'_\alpha = \nabla_\alpha \otimes \text{id} + \text{id} \otimes \nabla_\alpha^S$$

holds (in the notations of (2.1.2)).

An isomorphism is called flat if the corresponding isomorphism of gerbes ς is flat.

A connection on a twisted vector bundle is then a choice of connections on each descent datum of this twisted bundle and lifting of isomorphisms of descent data to flat isomorphisms of descent data with connections. So, every connection on a gerbe \mathcal{L} can be extended to a connection on any \mathcal{L} -twisted vector bundle.

Recall also the following propositions 2.22 and 2.23, cf. e.g. [4][pages 12-13].

Proposition 2.22. *Let \mathcal{E} be an \mathcal{L} -twisted bundle with connection. Choose a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ with a connection $(\nabla_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ representing \mathcal{E} . Then the collection $(\theta_\alpha + \omega_\alpha)$, where $\theta_\alpha = \nabla_\alpha^2$ is the curvature of ∇_α , defines a global differential 2-form θ on M with coefficients in the Azumaya bundle $\mathcal{A} = \text{End}(\mathcal{E})$. This form is independent of the choice of the representing descent datum.*

In the notations above let $\nabla: \mathcal{A} \rightarrow \Omega^1(M, \mathcal{A})$ be the connection defined for a fixed descent datum by

$$(2.2.3) \quad (\nabla \xi)|_{U_\alpha} = [\nabla_\alpha, \xi].$$

It is easy to see that ∇ is well defined and is a derivation with respect to the product on \mathcal{A} . Note that

$$(2.2.4) \quad \nabla^2 = [\theta, \cdot] \text{ and } \nabla \theta = \Omega$$

where Ω is the 3-curvature form of the connection on \mathcal{L} , see Lemma 2.11. In the following proposition, we use the notations of Appendix A, in particular the twisted differential d_Ω .

Proposition 2.23. *Let \mathcal{E} be an \mathcal{L} -twisted vector bundle and ∇ a connection on \mathcal{E} . Set $\text{Ch}_\mathcal{L}(\nabla) = \text{tr } e^{-u\theta} \in \Omega^*(M)[u]$. Then*

- (1) $d_\Omega \text{Ch}_\mathcal{L}(\nabla) = 0$.
- (2) *The class of $\text{Ch}_\mathcal{L}(\nabla)$ in $H_\Omega^*(M)$ is independent of choice of connections. Namely, assume that we are given a different connection ∇' on \mathcal{E} (and hence on \mathcal{L}) and let Ω' be the associated 3-curvature form. Then $I([\text{Ch}_\mathcal{L}(\nabla)]) = [\text{Ch}_\mathcal{L}(\nabla')]$, where I is the canonical isomorphism of $H_\Omega^*(M)$ with $H_{\Omega'}^*(M)$.*

Identifying as before the canonically isomorphic twisted cohomology spaces $H_\Omega^*(M)$, we denote the class of $\text{Ch}_\mathcal{L}(\nabla)$ by $\text{Ch}_\mathcal{L}(\mathcal{E})$.

Consider now an isomorphism S of twisted bundles $\mathcal{E}, \mathcal{E}'$ represented by the descent data as in Definition 2.21. Let θ and θ' be the curvature forms for \mathcal{E} and \mathcal{E}' respectively (cf. Proposition 2.22). Let $\mathcal{A}, \mathcal{A}'$ be the Azumaya bundles of \mathcal{E} and \mathcal{E}' and $S^*: \mathcal{A}' \rightarrow \mathcal{A}$ the induced isomorphism. Then an immediate consequence of (2.2.2) gives

$$S^*\theta' - \theta = c(\varsigma).$$

We will also need to use superconnections on twisted bundles and we now briefly indicate the modifications which need to be made to the notion of connection to obtain that of superconnection. Assume that we are given a gerbe \mathcal{L} and a \mathbb{Z}_2 -graded \mathcal{L} -twisted vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Let $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ be a descent datum for \mathcal{E} .

Definition 2.24. A superconnection \mathbb{A} on the descent datum is $\mathbb{A} = (\mathbb{A}_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ where $(\nabla_{\alpha\beta}, \omega_\alpha)$ is a connection on the descent datum for \mathcal{L} and each \mathbb{A}_α is a superconnection on \mathcal{E}_α satisfying the relations

$$\varphi_{\alpha\beta}^* \mathbb{A}_\beta = \mathbb{A}_\alpha \otimes \text{id} + \text{id} \otimes \nabla_{\alpha\beta}, \quad \text{for } U_{\alpha\beta} \neq \emptyset.$$

Each superconnection \mathbb{A}_α can be written as $\mathbb{A}_\alpha = \sum_{k \geq 0} \mathbb{A}_\alpha^{[k]}$ where $\mathbb{A}_\alpha^{[k]} \in \Omega^k(U_\alpha; \text{End}(\mathcal{E}_\alpha)^-)$ for k -even, $\mathbb{A}_\alpha^{[k]} \in \Omega^k(U_\alpha; \text{End}(\mathcal{E}_\alpha)^+)$ for k -odd, $k \neq 1$, and $\mathbb{A}_\alpha^{[1]}$ is a grading preserving connection on \mathcal{E}_α . It is easy to see that for each $k \neq 1$ there exists a form $\mathbb{A}^{[k]} \in \Omega^k(M; \text{End}(\mathcal{E}))$ such that $\mathbb{A}^{[k]}|_{U_\alpha} = \mathbb{A}_\alpha^{[k]}$. For $k = 1$, $(\mathbb{A}_\alpha^{[1]}, \nabla_{\alpha\beta}, \omega_\alpha)$ defines a connection on (the descent datum of) \mathcal{E} .

Let Ω be as before the curvature 3-form of the connection on \mathcal{L} . Define the curvature $\theta^{\mathbb{A}}$ of the superconnection \mathbb{A} by

$$\theta^{\mathbb{A}}|_{U_\alpha} = (\mathbb{A}_\alpha)^2 + \omega_\alpha.$$

We introduce a formal variable $u^{1/2}$ of degree -1 such that $(u^{1/2})^2 = u$ and define the new superconnection

$$\mathbb{A}_{u^{-1}} := \sum u^{(k-1)/2} \mathbb{A}^{[k]}.$$

Then the curvature $\theta^{\mathbb{A}_{u^{-1}}}$ of the superconnection $\mathbb{A}_{u^{-1}}$ is given by

$$\theta^{\mathbb{A}_{u^{-1}}}|_{U_\alpha} = (\mathbb{A}_{u^{-1}, \alpha})^2 + \omega_\alpha.$$

Notice then that $u\theta^{\mathbb{A}_{u^{-1}}} \in \Omega^{\text{even}}(M, \text{End}(\mathcal{E})^+)[u] + u^{1/2}\Omega^{\text{odd}}(M, \text{End}(\mathcal{E})^-)[u]$. We therefore deduce that

$$\exp(-u\theta^{\mathbb{A}_{u^{-1}}}) \in \Omega^{\text{even}}(M, \text{End}(\mathcal{E})^+)[u] + u^{1/2}\Omega^{\text{odd}}(M, \text{End}(\mathcal{E})^-)[u],$$

and the differential form $\text{str} \exp(-u\theta^{\mathbb{A}_{u^{-1}}})$ belongs to $\Omega^{\text{even}}(M)[u]$. The following is an analogue of Proposition 2.23 for the superconnections with the essentially identical proof.

Proposition 2.25.

- (1) Set $\text{Ch}_{\mathcal{L}}(\mathbb{A}) = \text{str} \exp(-u\theta^{\mathbb{A}_{u^{-1}}})$. Then $d_\Omega \text{Ch}_{\mathcal{L}}(\mathbb{A}) = 0$
- (2) The class of $\text{Ch}_{\mathcal{L}}(\mathbb{A})$ in $H_\Omega(M)$ is independent of choice of superconnection. Specifically, assume we are given a different superconnection \mathbb{A}' on \mathcal{E} (and therefore a different connection on \mathcal{L}) and let Ω' be the associated 3-curvature form. Then $I([\text{Ch}_{\mathcal{L}}(\mathbb{A})]) = [\text{Ch}_{\mathcal{L}}(\mathbb{A}')]$, where I is the canonical isomorphism of $H_\Omega^*(M)$ with $H_{\Omega'}^*(M)$.

Let $\pi : M \rightarrow B$ be a smooth fibration, then we need to extend the previous definitions to gerbes on M which are pulled back from B as follows. Let \mathcal{L} be a gerbe on B .

Definition 2.26.

- A descent datum for a transversally \mathcal{L} -twisted bundle \mathcal{E} on M consists of the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ for \mathcal{L} together with a collection $(\mathcal{E}_\alpha \rightarrow \pi^{-1}U_\alpha)_{\alpha \in \Lambda}$ of vector bundles and a collection of vector bundle isomorphisms $\varphi_{\alpha\beta} : \mathcal{E}_\alpha \otimes \pi^* \mathcal{L}_{\alpha\beta} \cong \mathcal{E}_\beta$ so that

$$(\pi^{-1}U_\alpha, \pi^* \mathcal{L}_{\alpha\beta}, \pi^* \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$$

is a descent datum for a twisted vector bundle on M .

- An (iso)morphism between two such descent data is given by the collection $(\rho_\alpha, S_\alpha, \lambda_{\alpha\beta})$ where $(S_\alpha, \lambda_{\alpha\beta})$ is an isomorphism between $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$ and $\rho_\alpha: \mathcal{E}_\alpha \otimes \pi^* S_\alpha \rightarrow \mathcal{E}'_\alpha$ is such that $(\rho_\alpha, \pi^* S_\alpha, \pi^* \lambda_{\alpha\beta})$ is an (iso)morphism in the usual sense between $(\pi^{-1}U_\alpha, \pi^* \mathcal{L}_{\alpha\beta}, \pi^* \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ and $(\pi^{-1}U'_\alpha, \pi^* \mathcal{L}'_{\alpha\beta}, \pi^* \mu'_{\alpha\beta\gamma}, \mathcal{E}'_\alpha, \varphi'_{\alpha\beta})$.

The notion of 2-morphism is defined similarly. A transversally twisted bundle is then an equivalence class of descent data.

Let $\text{Vect}_{\pi^* \mathcal{L}}(M)$ denote the set of isomorphism classes of all $\pi^* \mathcal{L}$ -twisted bundles on M and $\text{Vect}_{\pi^* \mathcal{L}}^\dagger(M)$ denote the set of isomorphism classes of all transversally \mathcal{L} -twisted bundles. Then according to Lemma 2.17, every class in $\text{Vect}_{\pi^* \mathcal{L}}(M)$ can be represented by a descent datum of a transversally \mathcal{L} -twisted bundle and the obvious forgetful map

$$\text{Vect}_{\pi^* \mathcal{L}}^\dagger(M) \longrightarrow \text{Vect}_{\pi^* \mathcal{L}}(M),$$

is thus surjective. Recall that (see [45], [46]) $\text{Vect}_{\pi^* \mathcal{L}}(M)$ is not empty if and only if the class $\pi^*[\mathcal{L}]$ is torsion in $H^3(M, \mathbb{Z})$. We then deduce immediately the following Proposition:

Proposition 2.27. *Transversely \mathcal{L} -twisted vector bundles exist on M if and only if the class $\pi^*[\mathcal{L}]$ is torsion in $H^3(M, \mathbb{Z})$.*

Notice that the above forgetful map is not injective. Indeed, if $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ is a descent datum for a transversally twisted bundle and S is a line bundle on M then $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha \otimes S|_{\pi^{-1}U_\alpha}, \varphi_{\alpha\beta} \otimes \text{id})$ is another such descent datum. These data define the same element of $\text{Vect}_{\pi^* \mathcal{L}}(M)$ but different elements of $\text{Vect}_{\pi^* \mathcal{L}}^\dagger(M)$, unless S is a pull-back of a line bundle from B .

A connection on the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ for a transversely \mathcal{L} -twisted bundle is a collection $(\nabla_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ where $(\nabla_{\alpha\beta}, \omega_\alpha)$ is a connection on the descent datum for \mathcal{L} and ∇_α is a connection on \mathcal{E}_α such that $(\nabla_\alpha, \pi^* \nabla_{\alpha\beta}, \pi^* \omega_\alpha)$ is a connection on the descent datum $(\pi^{-1}U_\alpha, \pi^* \mathcal{L}_{\alpha\beta}, \pi^* \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$. With these definitions one can now define a notion of connection on the transversally \mathcal{L} -twisted bundle in complete analogy with the definition for the twisted bundles. If ∇ is such a connection and Ω is the curvature 3-form of the gerbe \mathcal{L} , one defines $\text{Ch}_{\mathcal{L}}(\nabla)$, a closed form in $(\Omega^*(M), d_{\pi^* \Omega})$. The analogues of Propositions 2.22 and 2.23 hold in this context with the same proofs.

2.3. Projective families of pseudodifferential operators. We collect in this subsection several facts about families of pseudodifferential operators, associated with our submersion $M \rightarrow B$ and with transversely \mathcal{L} -twisted bundles. More details were given in [4] and we shall be brief. Let us start with the untwisted calculus.

We denote by $\Psi^m(M|B; E)$ the space of classical fiberwise pseudodifferential operators of order $\leq m$ on π , acting on the sections of the vector bundle E , which are fiberwise properly supported over M in the following sense. For $A \in \Psi^m(M|B; E)$ let $S_A \subset M \times M \times B$ be the support of the Schwartz kernel of A . We require that the maps $p_i: S_A \rightarrow X \times Y$, $i = 1, 2$ induced by the projections on the i th factor in $X \times X \times Y$ are proper. As usual, we set

$$\Psi(M|B; E) := \bigcup_{m \in \mathbb{Z}} \Psi^m(M|B; E) \text{ and } \Psi^{-\infty}(M|B; E) := \bigcap_{m \in \mathbb{Z}} \Psi^m(M|B; E).$$

Recall that composition endows each $\Psi(M|B; E)$ with the structure of a filtered algebra and a module over $C^\infty(B)$ such that the composition is $C^\infty(B)$ -linear and $\Psi^{-\infty}(M|B; E)$ is an ideal in this algebra. We have the following elementary general result which will be used in the sequel for different submersions:

Lemma 2.28. *Assume that $\pi: X \rightarrow Y$ is a smooth submersion, E is a vector bundle on X and L is a line bundle on Y . Define a map $\chi_L: \Psi(X|Y; E) \rightarrow \Psi(X|Y; E \otimes \pi^*L)$ by*

$$\chi_L(D)(e \otimes \pi^*(l)) = D(e) \otimes \pi^*l$$

for $D \in \Psi(X|Y; \mathcal{E})$, $e \in \Gamma_c(E)$, $l \in \Gamma(L)$. Then χ_L is a well-defined isomorphism of algebras and $C^\infty(Y)$ -modules. Moreover, we have for line bundles $L_1, L_2 \rightarrow Y$:

$$\chi_{L_1 \otimes L_2} = \chi_{L_2} \circ \chi_{L_1}.$$

Assume that $A \in \Psi(X|Y; E)$ is positive elliptic selfadjoint, and f is a Schwartz function. Then $f(\chi_L(A)) = \chi_L(f(A))$.

If we have two vector bundles E, E' on M we denote by $\Psi(M|B; E, E')$ the set of fiberwise pseudodifferential operators $\Gamma_c(E) \rightarrow \Gamma(E')$. For a line bundle L on B , we again have the isomorphism of $C^\infty(B)$ -modules $\chi_L: \Psi(M|B; E, E') \rightarrow \Psi(M|B; E \otimes \pi^*L, E' \otimes \pi^*L)$ defined by the same formula.

We have the vertical cotangent bundle $T^*(M|B) = T^*M/(\text{Ker } \pi_*)^\perp$. $\mathring{T}^*(M|B)$ denotes (the total space of) this bundle with the zero section removed, and $p: \mathring{T}^*(M|B) \rightarrow M$ is the natural projection. Recall that for $P \in \Psi^m(M|B; E, E')$ the principal symbol $\sigma_m(P)$ is a positively m -homogeneous smooth section over $\mathring{T}^*(M|B)$ of the vector bundle $p^* \text{Hom}(E, E')$. Then identifying the canonically isomorphic bundles $\text{Hom}(E, E')$ and $\text{Hom}(E \otimes \pi^*L, E' \otimes \pi^*L)$ we have $\sigma_m(P) = \sigma_m(\chi_L(P))$.

We now introduce the twist by a gerbe and assume that $\pi^*\mathcal{L}$ -twisted vector bundles exist on M , then we know that transversely \mathcal{L} -twisted bundles do exist as well, cf. Section 2.2. Let \mathcal{E} be such a transversally \mathcal{L} -twisted vector bundle on M . We fix a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ for \mathcal{E} . For any $(\alpha, \beta) \in \Lambda^2$ with $U_{\alpha\beta} \neq \emptyset$ we have an isomorphism of filtered algebras, respecting the $C^\infty(U_{\alpha\beta})$ -module structure:

$$(2.3.1) \quad \phi_{\alpha\beta}: \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\beta) \rightarrow \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\alpha),$$

It is defined as the composition

$$\Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\beta) \xrightarrow{\psi_{\alpha\beta}} \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\beta \otimes \pi^*\mathcal{L}_{\alpha\beta}) \xrightarrow{\varphi_{\alpha\beta}} \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\alpha)$$

where $\psi_{\alpha\beta} = \chi_{\mathcal{L}_{\alpha\beta}}^{-1} = \chi_{\mathcal{L}_{\beta\alpha}}$.

Recall (cf [7]) that for every $\alpha \in \Lambda$ we have an infinite dimensional bundle $\pi_*\mathcal{E}_\alpha$ on U_α defined by $\Gamma(V, \pi_*\mathcal{E}_\alpha) = \Gamma(\pi^{-1}V, \mathcal{E}_\alpha)$, $V \subset U_\alpha$. Over $U_{\alpha\beta}$ we have isomorphisms $\pi_*\varphi_{\alpha\beta}: \pi_*\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta} \rightarrow \pi_*\mathcal{E}_\beta$ defined by

$$\pi_*\varphi_{\alpha\beta}(\xi \otimes l) = \varphi(\xi \otimes \pi^*l).$$

Here $\xi \in \Gamma(U_{\alpha\beta}, \pi_*\mathcal{E}_\alpha) = \Gamma(\pi^{-1}U_{\alpha\beta}, \mathcal{E}_\alpha)$, $l \in \Gamma(U_{\alpha\beta}, \mathcal{L}_{\alpha\beta})$.

Note that the isomorphisms $\pi_*\varphi_{\alpha\beta}: \pi_*\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta} \rightarrow \pi_*\mathcal{E}_\beta$ induce the isomorphisms

$$(\pi_*\varphi_{\alpha\beta})^*: \text{End}(\pi_*\mathcal{E}_\beta) \rightarrow \text{End}(\pi_*\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta}) \cong \text{End}(\pi_*\mathcal{E}_\alpha)$$

over $U_{\alpha\beta}$. These isomorphisms restrict and yield the isomorphisms

$$\phi_{\alpha\beta}: \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}, \mathcal{E}_\beta) \longrightarrow \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}, \mathcal{E}_\alpha).$$

Since the isomorphisms $(\pi_*\varphi_{\alpha\beta})^*$ satisfy the natural cocycle identity we have the following:

Lemma 2.29. *The isomorphisms $\phi_{\alpha\beta}$ satisfy*

$$\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma} \quad \text{whenever } U_{\alpha\beta\gamma} \neq \emptyset.$$

Definition 2.30. A fiberwise pseudodifferential operator P of order $\leq m$ with coefficients in the transversally \mathcal{L} -twisted vector bundle \mathcal{E} is a collection $\{P_\alpha\}_{\alpha \in \Lambda}$, $P_\alpha \in \Psi^m(\pi^{-1}U_\alpha|U_\alpha; \mathcal{E}_\alpha)$ such that

$$P_\alpha = \phi_{\alpha\beta}(P_\beta).$$

where $\phi_{\alpha\beta}$ is defined in Equation 2.3.1. The space of fiberwise pseudodifferential operators of order $\leq m$, with coefficients in the transversely \mathcal{L} -twisted vector bundle \mathcal{E} , is denoted by $\Psi_{\mathcal{L}}^m(M|B; \mathcal{E})$.

Recall that the isomorphisms $\varphi_{\alpha\beta}$ induce the natural isomorphisms $\varphi_{\alpha\beta}^*: \text{End}(\mathcal{E}_\beta) \rightarrow \text{End}(\mathcal{E}_\alpha)$. Then

$$(2.3.2) \quad \sigma_m \circ \phi_{\alpha\beta} = p^* (\varphi_{\alpha\beta}^*) \circ \sigma_m$$

Equation (2.3.2) implies that if $P = \{P_\alpha\} \in \Psi_{\mathcal{L}}^m(M|B; \mathcal{E})$ then the collection $\sigma_m(P_\alpha)$ defines a section of the (untwisted) bundle $p^* \text{End}(\mathcal{E})$. We will call this section the principal symbol of $P = \{P_\alpha\}$ and denote it $\sigma_m(P)$.

We define in the same way the space $\Psi_{\mathcal{L}}^m(M|B; \mathcal{E}, \mathcal{E}')$ of fiberwise pseudodifferential operators of order $\leq m$, from the transversally \mathcal{L} -twisted vector bundle \mathcal{E} to the transversally \mathcal{L} -twisted vector bundle \mathcal{E}' . In particular $\Psi_{\mathcal{L}}^m(M|B; \mathcal{E}, \mathcal{E}) = \Psi_{\mathcal{L}}^m(M|B; \mathcal{E})$. We also have a principal symbol map

$$\sigma_m: \Psi_{\mathcal{L}}^m(M|B; \mathcal{E}, \mathcal{E}') \longrightarrow C^\infty(\mathring{T}^*(M|B), p^* \text{Hom}(\mathcal{E}, \mathcal{E}')).$$

We set

$$\Psi_{\mathcal{L}}(M|B; \mathcal{E}) := \bigcup_{m \in \mathbb{Z}} \Psi_{\mathcal{L}}^m(M|B; \mathcal{E}) \text{ and } \Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E}) := \bigcap_{m \in \mathbb{Z}} \Psi_{\mathcal{L}}^m(M|B; \mathcal{E}),$$

and introduce a composition in $\Psi_{\mathcal{L}}(M|B; \mathcal{E})$ by

$$\{P_\alpha\} \circ \{Q_\alpha\} = \{P_\alpha Q_\alpha\}$$

Since the maps $\phi_{\alpha\beta}$ are algebra isomorphisms the right hand side of this equality defines an element in $\Psi_{\mathcal{L}}(M|B; \mathcal{E})$. Recall that our operators are properly supported.

Proposition 2.31. *The composition of operators is $C^\infty(B)$ -linear and endows $\Psi_{\mathcal{L}}(M|B; \mathcal{E})$ with the structure of an algebra. Moreover, $\Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E})$ is a two-sided ideal in $\Psi_{\mathcal{L}}(M|B; \mathcal{E})$.*

Remark 2.32. *The construction can be sheafified, providing a sheaf of algebras on B , denoted by $\Psi_{\mathcal{L}}(\pi; \mathcal{E})$, and given by*

$$U \longmapsto \Psi_{\mathcal{L}}(\pi^{-1}U|U; \mathcal{E}), \text{ for open sets } U \subset B.$$

We define as usual the algebra of forms on B with values in $\Psi_{\mathcal{L}}(M|B; \mathcal{E})$ by

$$\Omega^*(B, \Psi_{\mathcal{L}}(M|B; \mathcal{E})) := \Omega^*(B) \otimes_{C^\infty(B)} \Psi_{\mathcal{L}}(M|B; \mathcal{E}).$$

Recall that for every α and $V \subset U_\alpha$ we have a fiberwise trace $\text{Tr}_\alpha: \Psi^{-\infty}(\pi^{-1}V|V; \mathcal{E}_\alpha) \rightarrow C^\infty(V)$. It is easy to see that for $P \in \Psi^{-\infty}(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\beta)$, we have $\text{Tr}_\alpha \phi_{\alpha\beta}(P) = \text{Tr}_\beta(P)$. We therefore obtain a well defined map

$$\text{Tr}: \Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E}) \rightarrow C^\infty(B) \text{ by setting } \text{Tr}\{P_\alpha\}|_{U_\alpha} = \text{Tr}_\alpha(P_\alpha).$$

This trace is a $C^\infty(B)$ -module map satisfying $\text{Tr}[A, B] = 0$. It extends naturally to define a map

$$\text{Tr}: \Omega^*(B, \Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E})) \longrightarrow \Omega^*(B).$$

If the bundle \mathcal{E} is \mathbb{Z}_2 graded we have a similarly defined supertrace

$$\text{Str}: \Omega^*(B, \Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E})) \longrightarrow \Omega^*(B).$$

Note that our definition of $\Psi_{\mathcal{L}}(M|B; \mathcal{E})$ depends on the descent datum for \mathcal{E} . It is straightforward however to see that an isomorphism of descent data defines canonically an isomorphism of the corresponding bundles of algebras. Moreover, every isomorphism $S: \mathcal{E} \rightarrow \mathcal{E}'$ of transversally twisted bundles with the underlying isomorphism ς between the corresponding gerbes \mathcal{L} and \mathcal{L}' and with the isomorphisms $\rho_\alpha: \mathcal{E}_\alpha \otimes S_\alpha \rightarrow \mathcal{E}'_\alpha$, induces an isomorphism of algebras

$$\Psi(S): \Omega^*(B, \Psi_{\mathcal{L}}(M|B; \mathcal{E})) \longrightarrow \Omega^*(B, \Psi_{\mathcal{L}'}(M|B; \mathcal{E}')).$$

We also get in the same way an isomorphism of the corresponding sheaves. Indeed, over any intersection $U_{\alpha\beta}$, the maps induced by the line bundles S_α and S_β and the isomorphisms ρ_α and ρ_β , obviously satisfy the relation

$$\phi'_{\alpha\beta} \circ \Psi(\rho_\beta) = \Psi(\rho_\alpha) \circ \phi_{\alpha\beta}.$$

They thus define the isomorphism $\Psi(S)$. The following is then an easy consequence of the definitions.

Lemma 2.33.

- 1] Let $S: \mathcal{E} \rightarrow \mathcal{E}'$, $S': \mathcal{E} \rightarrow \mathcal{E}'$ be two isomorphisms of transversally twisted bundles. If there exists a 2-morphism $S \Rightarrow S'$ then $\Psi(S) = \Psi(S')$.
- 2] Let $S_1: \mathcal{E} \rightarrow \mathcal{E}'$, $S_2: \mathcal{E}' \rightarrow \mathcal{E}''$ be two isomorphisms of transversally twisted bundles. Then $\Psi(S_2 \circ S_1) = \Psi(S_2) \circ \Psi(S_1)$.
- 3] For any $P \in \Omega^*(B, \Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E}))$, $\text{Str } \Psi(S)(P) = \text{Str } P$.

2.4. Connections and curvatures. We continue in the notations of the previous paragraph. We assume that we are given a transversally \mathcal{L} -twisted bundle \mathcal{E} with connection represented by a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ with connections $(\nabla_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$, see Subsection 2.2.

Recall the bundles $\pi_*\mathcal{E}_\alpha$ and isomorphisms $\pi_*\varphi_{\alpha\beta}$ defined in Subsection 2.3. It is easy to see that $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \pi_*\mathcal{E}_\alpha, \pi_*\varphi_{\alpha\beta})$ is a descent datum for an infinite dimensional twisted bundle. We now proceed to define a connection on this descent datum.

Choose a horizontal distribution i.e. a subbundle $\mathcal{H} \subset TM$ such that $TM = \mathcal{H} \oplus T(M|B)$. This choice together with connections $\nabla_\alpha^\mathcal{E}$ defines for each α a connection $\nabla_\alpha^\mathcal{H}$ on $\pi_*\mathcal{E}_\alpha$ as follows:

$$(\nabla_\alpha^\mathcal{H})_X \xi = (\nabla_\alpha^\mathcal{E})_{X^\mathcal{H}} \xi$$

where $X^\mathcal{H}$ is the horizontal lift of $X \in \Gamma(B, TB)$.

Lemma 2.34. $(\pi_*\varphi_{\alpha\beta})^* \nabla_\beta^\mathcal{H} = \nabla_\alpha^\mathcal{H} \otimes \text{id} + \text{id} \otimes \nabla_{\alpha\beta}$

The curvature of the connection $\nabla_\alpha^\mathcal{H}$ is a 2-form $\theta_\alpha^\mathcal{H}$ on U_α with values in fiberwise first order differential operators given by

$$\theta_\alpha^\mathcal{H}(X, Y) = \theta_\alpha^\mathcal{E}(X^\mathcal{H}, Y^\mathcal{H}) + (\nabla_\alpha^\mathcal{E})_{T^\mathcal{H}(X, Y)}.$$

where

$$(2.4.1) \quad T^\mathcal{H}(X, Y) = [X^\mathcal{H}, Y^\mathcal{H}] - [X, Y]^\mathcal{H}, \quad X, Y \in \Gamma(B, TB).$$

Each $\nabla_\alpha^\mathcal{H}$ defines a filtration-preserving derivation $\partial_\alpha^\mathcal{H}$ of the algebra of fiberwise pseudodifferential operators in the following way:

$$\begin{aligned} \partial_\alpha^\mathcal{H}: \Psi(\pi^{-1}U_\alpha|U_\alpha, \mathcal{E}_\alpha) &\longrightarrow \Omega^1(U_\alpha, \Psi(\pi^{-1}U_\alpha|U_\alpha, \mathcal{E}_\alpha)) \\ D &\longmapsto \partial_\alpha^\mathcal{H}(D) = [\nabla_\alpha^\mathcal{H}, D]. \end{aligned}$$

If $D \in \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}, \mathcal{E}_\beta)$ then the result of Lemma 2.34 implies that

$$\partial_\alpha^\mathcal{H}(\phi_{\alpha\beta}(D)) = \phi_{\alpha\beta}(\partial_\beta^\mathcal{H}(D)).$$

Therefore if $\{D_\alpha\}$, $D_\alpha \in \Psi_\mathcal{L}(\pi^{-1}U_\alpha|U_\alpha, \mathcal{E}_\alpha)$, defines an element in $\Psi_\mathcal{L}^m(M|B, \mathcal{E})$ then $\{\partial_\alpha^\mathcal{H}(D_\alpha)\}$ defines an element of $\Omega^1(B, \Psi_\mathcal{L}^m(M|B, \mathcal{E}))$. We therefore obtain a derivation

$$(2.4.2) \quad \partial^\mathcal{H}: \Psi_\mathcal{L}(M|B, \mathcal{E}) \longrightarrow \Omega^1(B, \Psi_\mathcal{L}(M|B, \mathcal{E})),$$

which extends to a derivation of the algebra $\Omega^*(B, \Psi_\mathcal{L}(M|B, \mathcal{E}))$.

Lemma 2.35. *There exists $\theta^\mathcal{H} \in \Omega^2(B, \Psi_\mathcal{L}^1(M|B, \mathcal{E}))$ such that*

$$(2.4.3) \quad \theta^\mathcal{H}|_{U_\alpha} = \theta_\alpha^\mathcal{H} + \omega_\alpha$$

Proof. By Lemma 2.34, $\phi_{\alpha\beta}^* \theta_\beta^\mathcal{H} = \theta_\alpha^\mathcal{H} + \pi^*(\omega_\alpha - \omega_\beta)$, and the statement follows as in Proposition 2.22. \square

We have

$$(\partial^\mathcal{H})^2(D) = [\theta^\mathcal{H}, D] \text{ and } \partial^\mathcal{H}(\theta^\mathcal{H}) = \pi^*\Omega.$$

where Ω is the 3-curvature form of the connection on \mathcal{L} .

Assume that S is an isomorphism of transversally twisted bundles \mathcal{E} and \mathcal{E}' on M and ς is the underlying isomorphism of the underlying gerbes $\mathcal{L}, \mathcal{L}'$ respectively on B . Recall that it induces an isomorphism $\Psi(S): \Psi_{\mathcal{L}}(M|B, \mathcal{E}) \rightarrow \Psi_{\mathcal{L}'}(M|B, \mathcal{E}')$.

Proposition 2.36. *The following identities hold:*

- (1) $(\partial^{\mathcal{H}})' \circ \Psi(S) = \Psi(S) \circ \partial^{\mathcal{H}}$.
- (2) $\Psi(S)\theta^{\mathcal{H}} - (\theta^{\mathcal{H}})' = c(\varsigma)$.

3. GERBES ON GROUPOIDS

3.1. \mathcal{G} -Gerbes and their morphisms. Given a gerbe \mathcal{L} on $\mathcal{G}^{(0)}$, we get two gerbes $s^*\mathcal{L}$ and $r^*\mathcal{L}$ over the manifold $\mathcal{G}^{(1)}$ by pulling back \mathcal{L} under the source and range maps s and r .

Definition 3.1. A gerbe $(\mathcal{L}, \varsigma, \nu)$ on \mathcal{G} , is given by the following data:

- A gerbe \mathcal{L} on the unit space $\mathcal{G}^{(0)}$.
- An isomorphism $\varsigma: s^*\mathcal{L} \rightarrow r^*\mathcal{L}$ of gerbes on $\mathcal{G}^{(1)}$.
- A 2-morphism $\nu: \varsigma_{01}^2 \circ \varsigma_{12}^2 \rightarrow \varsigma_{02}^2$ such that the following diagram commutes:

$$\begin{array}{ccc} \varsigma_{01}^3 \circ \varsigma_{12}^3 \circ \varsigma_{23}^3 & \xrightarrow{\text{id} \otimes \nu_{012}} & \varsigma_{02}^3 \circ \varsigma_{23}^3 \\ \nu_{123} \otimes \text{id} \downarrow & & \downarrow \nu_{023} \\ \varsigma_{01}^3 \circ \varsigma_{13}^3 & \xrightarrow{\nu_{013}} & \varsigma_{03}^3 \end{array}$$

In the previous definition, we have used the notion of 2-morphism of definition 2.3 and we have denoted $\varsigma_{ij}^k = (\text{pr}_{ij}^k)^*\varsigma$ and $\nu_{ij\ell} = \nu_{ij\ell}^3 = (\text{pr}_{ij\ell}^3)^*\nu$. We shall also refer to a gerbe $(\mathcal{L}, \varsigma, \nu)$ on \mathcal{G} as a \mathcal{G} -gerbe.

Definition 3.2. A morphism between two gerbes $(\mathcal{L}, \varsigma, \nu), (\mathcal{L}', \varsigma', \nu')$ on \mathcal{G} is given by the data (u, ϱ) where:

- $u: \mathcal{L} \rightarrow \mathcal{L}'$ is a morphism of gerbes over $\mathcal{G}^{(0)}$.
- (ϱ, s^*u, r^*u) is a 2-morphism between ς and ς' in the sense of Definition 2.3, which is compatible with the 2-morphisms ν and ν' over $\mathcal{G}^{(2)}$.

The compatibility condition with the 2-morphisms ν and ν' over $\mathcal{G}^{(2)}$ means that the following relation holds (with the obvious notations)

$$\varrho_{02}^2 \circ (\text{id} \otimes \nu) = (\nu' \otimes \text{id}) \circ (\text{id} \otimes \varrho_{01}^2) \circ (\varrho_{12}^2 \otimes \text{id}).$$

Notice that $s^*u: s^*\mathcal{L} \rightarrow s^*\mathcal{L}'$ and $r^*u: r^*\mathcal{L} \rightarrow r^*\mathcal{L}'$ are the pull-back morphisms and that $(s^*u)_{01} = (r^*u)_{12}$, $(r^*u)_{01} = (r^*u)_{02}$ and $(s^*u)_{12} = (s^*u)_{02}$. The compatibility is hence equivalent to the commutativity of the following diagram

$$\begin{array}{ccc} \varsigma'_{01} \circ (r^*u)_{12} \circ \varsigma_{12} & \xrightarrow{\varrho_{12} \otimes \text{id}} & \varsigma'_{01} \circ \varsigma'_{12} \circ (s^*u)_{12} \\ \text{id} \otimes \varrho_{01}^{-1} \downarrow & & \downarrow \text{id} \otimes \nu' \\ (r^*u)_{01} \circ \varsigma_{01} \circ \varsigma_{12} & \xrightarrow{\varrho_{02} \circ (\nu \otimes \text{id})} & \varsigma'_{02} \circ (s^*u)_{02} \end{array}$$

Definition 3.3. Consider two gerbes $(\mathcal{L}, \varsigma, \nu)$, $(\mathcal{L}', \varsigma', \nu')$ on \mathcal{G} and two isomorphisms (u, ϱ) , (u', ϱ') between them. A 2-morphism between them is given by a 2-morphism $\beta: u \rightarrow u'$, such that

$$\varrho' \circ (s^* \beta \otimes \text{id}) = (\text{id} \otimes r^* \beta) \circ \varrho.$$

We now comment briefly on the case when the gerbe \mathcal{L} on $\mathcal{G}^{(0)}$ is trivialized. The morphism ς in this case is a line bundle on $\mathcal{G}^{(1)}$. The gerbe L therefore is given by the following data (cf. Definition 3.1):

- A line bundle on $\mathcal{G}^{(1)}$.
- an isomorphism of line bundles $\nu: \varsigma_{01}^2 \circ \varsigma_{12}^2 \rightarrow \varsigma_{02}^2$ such that the following diagram commutes:

$$\begin{array}{ccc} \varsigma_{01}^3 \circ \varsigma_{12}^3 \circ \varsigma_{23}^3 & \xrightarrow{\text{id} \otimes \nu_{012}} & \varsigma_{02}^3 \circ \varsigma_{23}^3 \\ \nu_{123} \otimes \text{id} \downarrow & & \downarrow \nu_{023} \\ \varsigma_{01}^3 \circ \varsigma_{13}^3 & \xrightarrow{\nu_{013}} & \varsigma_{03}^3 \end{array}$$

One can construct now a convolution algebra of \mathcal{G} twisted by L . As a vector space, it coincides with the compactly supported sections of ς , $C_c^\infty(\mathcal{G}^{(1)}, \varsigma)$. The convolution product is given, for $s_1, s_2 \in C_c^\infty(\mathcal{G}^{(1)}, \varsigma)$ by

$$s_1 s_2 = (\text{pr}_{02}^2)_* (\nu((\text{pr}_{02}^2)^* s_1 \otimes (\text{pr}_{02}^2)^* s_2)),$$

i.e.

$$s_1 s_2(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} \nu(s_1(\gamma_1) \otimes s_2(\gamma_2)).$$

3.2. Connections on \mathcal{G} -gerbes.

A connection on a \mathcal{G} -gerbe $(\mathcal{L}, \varsigma, \nu)$ is given by

- A choice of connection $(\nabla^{\mathcal{L}}, \omega^{\mathcal{L}})$ on \mathcal{L}
- A choice of connection ∇^ς on $\varsigma: s^* \mathcal{L} \rightarrow r^* \mathcal{L}$ so that it becomes a morphism of gerbes with connective structures.

Then the constructions given in Subsection 2.1 can be applied to the gerbe \mathcal{L} on $\mathcal{G}^{(0)}$, the morphism ς of gerbes with connection $r^* \mathcal{L}$, $s^* \mathcal{L}$ on $\mathcal{G}^{(1)}$ and the 2-morphism $\nu: \varsigma_{01} \circ \varsigma_{12} \rightarrow \varsigma_{02}$ on $\mathcal{G}^{(2)}$ to show the existence of connections on \mathcal{G} -gerbes and to give the following differential forms:

$$\Omega(\mathcal{L}) \in \Omega^3(\mathcal{G}^{(0)}), \quad c(\varsigma) \in \Omega^2(\mathcal{G}^{(1)}), \quad c(\nu) \in \Omega^1(\mathcal{G}^{(2)})$$

We introduce now the following filtration on $\Omega^\bullet(\mathcal{G}^{(\bullet)})$

$$F^k \Omega^\bullet(\mathcal{G}^{(\bullet)}) = \bigoplus_{k+i \geq 0} \Omega^i(\mathcal{G}^{(\bullet)}).$$

Lemma 3.4. *The degree 3 element $(\Omega(\mathcal{L}), c(\varsigma), c(\nu))$ in $F^{-1} \Omega^\bullet(\mathcal{G}^{(\bullet)})$ is closed. Its class in the cohomology of $F^{-1} \Omega^\bullet(\mathcal{G}^{(\bullet)})$ is independent of choices made (connection on \mathcal{L} and lift of ς).*

Proof. A straightforward computation gives:

$$d\Omega(\mathcal{L}) = 0, \quad \partial_0 \Omega(\mathcal{L}) - dc(\varsigma) = 0, \quad \partial_0 c(\varsigma) - \partial_1 c(\varsigma) + dc(\nu) = 0 \quad \text{and} \quad \partial c(\nu) = 0.$$

Hence we see that $(d + \partial)(\Omega(\mathcal{L}) + c(\varsigma) + c(\nu)) = 0$. The second part of the lemma is a consequence of Proposition 2.12. \square

Definition 3.5. Associated with any \mathcal{G} gerbe, there is a well defined DD class in the cohomology of the subcomplex $F^{-1}\Omega^\bullet(\mathcal{G}^{(\bullet)})$, namely the cohomology class of the 3-cocycle

$$\frac{1}{2\pi i}(\Omega(\mathcal{L}), c(\varsigma), c(\nu)).$$

Remark 3.6. Recall that the complexes $\Omega^\bullet(\mathcal{G}^{(\bullet)})$ and $\Omega_{DS}^\bullet(\mathcal{G})$ are quasi-isomorphic. An explicit formula for the quasi-isomorphism $I_\Delta: \Omega_{DS}^\bullet(\mathcal{G}) \rightarrow \Omega^\bullet(\mathcal{G}^{(\bullet)})$ can be found in [31], Theorem 6.4.

To end this paragraph, let us recall that a \mathcal{G} -sheaf is a sheaf on $\mathcal{G}^{(0)}$ such that for every $\gamma \in \mathcal{G}$ there is morphism $\mathcal{A}_{s(\gamma)} \rightarrow \mathcal{A}_{r(\gamma)}$, satisfying the standard action axioms. Here and as usual \mathcal{A}_x is the stalk of \mathcal{A} at x . Examples are for instance the \mathcal{G} -sheaves $C_\mathcal{G}^\infty$ and $(C_\mathcal{G}^\infty)^*$ corresponding respectively to smooth and smooth nonvanishing functions.

Remark 3.7. Similar to the case of gerbes on manifolds, one associates with any \mathcal{G} gerbe as above, its class in $H^2(\mathcal{G}, (C_\mathcal{G}^\infty)^*)$. The exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow C_\mathcal{G}^\infty \rightarrow (C_\mathcal{G}^\infty)^* \rightarrow 0$$

then gives rise to an exact sequence

$$\dots \rightarrow H^2(\mathcal{G}, C_\mathcal{G}^\infty) \rightarrow H^2(\mathcal{G}, (C_\mathcal{G}^\infty)^*) \rightarrow H^3(\mathcal{G}, \mathbb{Z}) \rightarrow H^3(\mathcal{G}, C_\mathcal{G}^\infty) \rightarrow \dots$$

However the cohomology groups $H^\bullet(\mathcal{G}, C_\mathcal{G}^\infty)$ in positive degree do not vanish in general, and the map $H^2(\mathcal{G}, (C_\mathcal{G}^\infty)^*) \rightarrow H^3(\mathcal{G}, \mathbb{Z})$ is therefore not an isomorphism. It is an isomorphism, however, if \mathcal{G} is proper, see e.g. Lemma 1.1.

Remark 3.8. The morphism of sheaves $d\log: (C_\mathcal{G}^\infty)^* \rightarrow (\Omega_\mathcal{G}^1)^{cl}$, where $(\Omega_\mathcal{G}^1)^{cl}$ is the sheaf of closed 1-forms, induces a map $H^\bullet(\mathcal{G}, (C_\mathcal{G}^\infty)^*) \rightarrow H^\bullet(\mathcal{G}, (\Omega_\mathcal{G}^1)^{cl})$. Using the resolution $(\Omega_\mathcal{G}^{\geq 1}, d_{dR})$ of $(\Omega_\mathcal{G}^1)^{cl}$, we obtain that the latter cohomology is computed by the complex $F^{-1}\Omega^\bullet(\mathcal{G}^{(\bullet)})[1]$. We therefore obtain a map $H^2(\mathcal{G}, (C_\mathcal{G}^\infty)^*) \rightarrow H^3(F^{-1}\Omega^\bullet(\mathcal{G}^{(\bullet)}))$. The class given in Definition 3.5 is the image of the class introduced in the Remark 3.7 under this map. However we will not use this fact, since we only use the class from Corollary 3.5.

3.3. DD class in the Dupont-Sullivan bicomplex. We now describe explicitly a representative (a 3-cocycle $\Omega(L)$ in the Dupont-Sullivan bicomplex) of the canonical DD class associated with the \mathcal{G} gerbe L satisfying

$$I_\Delta \left(\left[\frac{\Omega(L)}{2i\pi} \right] \right) = \left[\frac{1}{2i\pi} (\Omega(\mathcal{L}), c(\varsigma), c(\nu)) \right].$$

The notation should be clear and no confusion should occur between this 3-cocycle $\Omega(L)$ and the differential 3-form $\Omega(\mathcal{L})$ associated with the gerbe \mathcal{L} on $\mathcal{G}^{(0)}$. The components of $\Omega(L)$ are given in Lemma 3.10 below.

Assume then that $L = (\mathcal{L}, \varsigma, \nu)$ is a gerbe with connection on \mathcal{G} . Let $\mathcal{L}^{(k)}$ be the pullback of \mathcal{L} to $\mathcal{G}^{(k)}$ via the map $\text{pr}_0^k: (\gamma_1, \dots, \gamma_k) \mapsto r(\gamma_1)$. There are isomorphisms

$$(\text{pr}_{0i}^k)^* \varsigma: (\text{pr}_i^k)^* \mathcal{L} \rightarrow \mathcal{L}^{(k)}$$

of gerbes with connective structures, obtained by using the pulled-back connections. Since $\text{pr}_0^{k-1} \circ \partial_i = \text{pr}_0^k$, for $i \geq 1$, and $\text{pr}_0^{k-1} \circ \partial_0 = \text{pr}_1^k$, we have canonical isomorphisms of gerbes with connections

$$(3.3.1) \quad \partial_i^* \mathcal{L}^{(k-1)} \cong \mathcal{L}^{(k)}, \quad \text{for } i \geq 1,$$

and a morphism of gerbes

$$(3.3.2) \quad (\text{pr}_{01}^k)^* \varsigma: \partial_0^* \mathcal{L}^{(k-1)} \rightarrow \mathcal{L}^{(k)}, \quad \text{for } i = 0.$$

Both of the gerbes in the equation (3.3.2) are equipped with connections, and $(\text{pr}_{01}^k)^* \varsigma$ becomes a morphism of gerbes with connective structures when equipped with any of the connections ∇_i determined by the condition that the 2-morphism

$$(\text{pr}_{01i}^k)^* \nu: (\text{pr}_{01}^k)^* \varsigma \circ (\text{pr}_{1i}^k)^* \varsigma \rightarrow (\text{pr}_{0i}^k)^* \varsigma$$

is flat. Here the left hand side is equipped with the connection $\nabla_i \otimes (\text{pr}_{1i}^k)^* \nabla^\varsigma$ and the right hand side is equipped with the connection $(\text{pr}_{0i}^k)^* \nabla^\varsigma$. For simplicity, we omit from the notations the pullback to the simplex Δ^k .

We have

$$\nabla_1 = (\text{pr}_{01}^k)^* \nabla^\varsigma \text{ and } \nabla_i - (\text{pr}_{01}^k)^* \nabla^\varsigma = c((\text{pr}_{01i}^k)^* \nu).$$

In addition to its curving $\omega_0 := (\text{pr}_0^k)^* \omega^\mathcal{L}$, the gerbe $\mathcal{L}^{(k)}$ has curvings

$$\omega_i := ((\text{pr}_{0i}^k)^* \varsigma^{-1})^* (\text{pr}_i^k)^* \omega^\mathcal{L} \quad \text{for any } i = 1, \dots, k$$

which are also compatible with the connective structure, and we have

$$\omega_i - \omega_j = c((\text{pr}_{ij}^k)^* \varsigma).$$

Let $\mathcal{L}_\Delta^{(k)}$ denote the pull-back of $\mathcal{L}^{(k)}$ to $\mathcal{G}^{(k)} \times \Delta^k$ via the projection on the first factor. A direct calculation shows that we have for any $i \geq 1$ flat isomorphisms of gerbes with connections

$$(3.3.3) \quad (\partial_i \times \text{id})^* \mathcal{L}_\Delta^{(k-1)} \cong (\text{id} \times \delta_i)^* \mathcal{L}_\Delta^{(k)}$$

induced by (3.3.1).

For $i = 0$ we have an isomorphism of gerbes

$$(3.3.4) \quad (\text{pr}_{01}^k)^* \varsigma: (\partial_0 \times \text{id})^* \mathcal{L}_\Delta^{(k-1)} \rightarrow (\text{id} \times \delta_0)^* \mathcal{L}_\Delta^{(k)}$$

Equip now $\mathcal{L}_\Delta^{(k)}$ (for every k) with connection $(\nabla_\Delta, \omega_\Delta)$ where

$$(3.3.5) \quad \nabla_\Delta = (\text{pr}_0^k)^* \nabla^\mathcal{L}, \text{ and}$$

$$(3.3.6) \quad \omega_\Delta := \sum_{i=0}^k t_i \omega_i - \sum_{0 < i < j \leq k} (t_i dt_j - t_j dt_i) (\text{pr}_{0,i,j}^k)^* c(\nu) =$$

$$\omega_0 - \sum_{i=1}^k t_i (\text{pr}_{0,i}^k)^* c(\varsigma) - \sum_{0 < i < j \leq k} (t_i dt_j - t_j dt_i) (\text{pr}_{0,i,j}^k)^* c(\nu)$$

Equip finally the isomorphism (3.3.4) with the connection

$$(3.3.7) \quad \nabla_\Delta^\varsigma := \sum_{i=1}^k t_i \nabla_i = (\text{pr}_{0,1}^k)^* \nabla^\varsigma + \sum_{i=2}^k t_i c((\text{pr}_{0,1,i}^k)^* \nu)$$

Lemma 3.9. *With the choice of connections and curvings given in (3.3.5), (3.3.6) and (3.3.7), the maps given by (3.3.3) and (3.3.4) are flat isomorphisms of gerbes with connections.*

Proof. Recall that by the definition of flatness (cf. Lemma 2.8) we need to verify that the curvatures of the isomorphisms in (3.3.3) (for $i \geq 1$) and (3.3.4) (for $i = 0$) vanish. The proof for $i \geq 1$ is straightforward; we verify the statement for $i = 0$. It is clear that (3.3.4) is a morphism of gerbes with connective structures, so we just need to verify its flatness. If we equip the left and right hand sides with curvings ω_1 and ω_0 respectively and (3.3.4) with the connection $(\text{pr}_{0,1}^k)^* \nabla^\varsigma$, the curvature of (3.3.4) is $(\text{pr}_{0,i}^k)^* c(\varsigma)$. It follows that if we use curvings given by (3.3.6) and connection given by (3.3.7) the curvature of (3.3.4) will be (we denote coordinates on Δ^{k-1} by t_1, t_2, \dots, t_k with $t_1 + t_2 + \dots + t_k = 1$)

$$(3.3.8) \quad (\text{pr}_{0,1}^k)^* c(\varsigma) - \sum_{i=1}^k t_i (\text{pr}_{0,i}^k)^* c(\varsigma) - \sum_{0 < i < j \leq k} (t_i dt_j - t_j dt_i) (\text{pr}_{0,i,j}^k)^* c(\nu) +$$

$$\sum_{i=1}^k t_i (\text{pr}_{1,i}^k)^* c(\varsigma) + \sum_{1 < i < j \leq k} (t_i dt_j - t_j dt_i) (\text{pr}_{1,i,j}^k)^* c(\nu) + d \left(\sum_{i=2}^k t_i c((\text{pr}_{0,1,i}^k)^* \nu) \right) =$$

$$\sum_{i=2}^k t_i ((\text{pr}_{0,1}^k)^* c(\varsigma) + (\text{pr}_{1,i}^k)^* c(\varsigma) - (\text{pr}_{0,i}^k)^* c(\varsigma)) +$$

$$\sum_{0 < i < j \leq k} (t_i dt_j - t_j dt_i) ((\text{pr}_{1,i,j}^k)^* c(\nu) - (\text{pr}_{0,i,j}^k)^* c(\nu)) + d \left(\sum_{i=2}^k t_i c((\text{pr}_{0,1,i}^k)^* \nu) \right).$$

Using the identities

$$(\text{pr}_{0,1}^k)^* c(\varsigma) + (\text{pr}_{1,i}^k)^* c(\varsigma) - (\text{pr}_{0,i}^k)^* c(\varsigma) = -d(\text{pr}_{0,1,i}^k)^* c(\nu)$$

$$(\text{pr}_{1,i,j}^k)^* c(\nu) - (\text{pr}_{0,i,j}^k)^* c(\nu) = -(\text{pr}_{0,1,j}^k)^* c(\nu) + (\text{pr}_{0,1,i}^k)^* c(\nu)$$

we rewrite the sum (3.3.8) as

$$(3.3.9) \quad \sum_{0 < i < j \leq k} (t_i dt_j - t_j dt_i) (-(\text{pr}_{0,1,j}^k)^* c(\nu) + (\text{pr}_{0,1,i}^k)^* c(\nu)) + \sum_{i=2}^k dt_i c((\text{pr}_{0,1,i}^k)^* \nu).$$

But notice that

$$\begin{aligned} \sum_{0 < i < j \leq k} (t_i dt_j - t_j dt_i) \left(-(\text{pr}_{0,1,j}^k)^* c(\nu) + (\text{pr}_{0,1,i}^k)^* c(\nu) \right) = \\ \sum_{0 < i, j \leq k} (t_i dt_j - t_j dt_i) (\text{pr}_{0,1,i}^k)^* c(\nu) = - \sum_{0 < i \leq k} dt_i (\text{pr}_{0,1,i}^k)^* c(\nu). \end{aligned}$$

Whence we conclude that the sum in the (3.3.9) (and thus the sum in (3.3.8)) is equal to 0. This verifies the flatness of the morphism (3.3.4). \square

Lemma 3.10. *With the choice of connection as above*

$$\Omega(\mathcal{L}_\Delta^{(k)}) = \sum_{i=0}^k t_i (\text{pr}_i^k)^* \Omega(\mathcal{L}) - \sum_{0 < i < j \leq k} (t_i dt_j - t_j dt_i) (\text{pr}_{ij}^k)^* c(\varsigma) - 2 \sum_{0 < i < j \leq k} dt_i dt_j (\text{pr}_{0ij}^k)^* c(\nu).$$

Proof. We use the identities

$$\begin{aligned} c((\text{pr}_{0i}^k)^* \varsigma) &= \omega_0 - \omega_i \\ d(\text{pr}_{0,i,j}^k)^* c(\nu) &= -(\text{pr}_{0,i}^k)^* c(\varsigma) - (\text{pr}_{i,j}^k)^* c(\varsigma) + (\text{pr}_{0,j}^k)^* c(\varsigma) \end{aligned}$$

to compute

$$\begin{aligned} \Omega(\mathcal{L}_\Delta^{(k)}) &= \sum_{i=0}^k t_i (\text{pr}_i^k)^* \Omega(\mathcal{L}) - \sum_{i=1}^k dt_i (\text{pr}_{0,i}^k)^* c(\varsigma) - 2 \sum_{0 < i < j \leq k} dt_i dt_j (\text{pr}_{0,i,j}^k)^* c(\nu) + \\ &\quad \sum_{0 < i < j \leq k} (t_i dt_j - t_j dt_i) \left(-(\text{pr}_{0,i}^k)^* c(\varsigma) - (\text{pr}_{i,j}^k)^* c(\varsigma) + (\text{pr}_{0,j}^k)^* c(\varsigma) \right). \end{aligned}$$

Now, the statement follows since

$$\begin{aligned} \sum_{0 < i < j \leq k} (t_i dt_j - t_j dt_i) \left(-(\text{pr}_{0,i}^k)^* c(\varsigma) + (\text{pr}_{0,j}^k)^* c(\varsigma) \right) = \\ \sum_{0 < i, j \leq k} (t_i dt_j - t_j dt_i) (\text{pr}_{0,j}^k)^* c(\varsigma) = \sum_{0 < j \leq k} dt_j (\text{pr}_{0,j}^k)^* c(\varsigma). \end{aligned}$$

\square

Definition 3.11. The collection of forms $\Omega(\mathcal{L}_\Delta^{(k)})$ defines the element $\Omega(L) \in \Omega_{DS}^3(\mathcal{G})$, with vanishing $\Omega_{DS}^{0,3}(\mathcal{G})$ component.

We therefore can form, as in Subsection 1.2, Equation (1.2.1), a twisted differential $d_{\Omega(L)}^t$ and consider the twisted cohomology $H_{\Omega(L)}^\bullet(B\mathcal{G})$ of the classifying space $B\mathcal{G}$.

4. PROJECTIVE \mathcal{G} -FAMILIES AND THE ANALYTIC INDEX

4.1. Twisted vector bundles on groupoids. Assume that $L = (\mathcal{L}, \varsigma, \nu)$ is a \mathcal{G} -gerbe as before.

Definition 4.1. An L -twisted vector bundle $E = (\mathcal{E}, \phi)$ is given by the following data:

- (1) An \mathcal{L} -twisted vector bundle \mathcal{E} on $\mathcal{G}^{(0)}$.

- (2) An isomorphism of twisted vector bundles $\phi: s^*\mathcal{E} \rightarrow r^*\mathcal{E}$ such that:
- The underlying isomorphism of gerbes is $\varsigma: s^*\mathcal{L} \rightarrow r^*\mathcal{L}$.
 - The 2-morphism $\nu: \varsigma_{01}^2 \circ \varsigma_{12}^2 \rightarrow \varsigma_{02}^2$ defines a 2-morphism $\nu: \phi_{01}^2 \circ \phi_{12}^2 \rightarrow \phi_{02}^2$.

We have used Definition 2.15 and denoted as before $\phi_{ij}^2 := (\pi_{ij}^2)^*\phi$. Assume that $E = (\mathcal{E}, \phi)$ is an L -twisted vector bundle and let $\nabla^\mathcal{E}$ be a connection on \mathcal{E} . As for the gerbe, we denote by $\mathcal{E}^{(k)}$ the pullback of \mathcal{E} to $\mathcal{G}^{(k)}$ via the map pr_0^k .

Note also that there is an isomorphism $(\text{pr}_{0i}^k)^*\phi: (\text{pr}_i^k)^*\mathcal{E} \rightarrow \mathcal{E}^{(k)}$ and similarly to (3.3.1), (3.3.2) we as well have canonical isomorphisms

$$(4.1.1) \quad \partial_i^* \mathcal{E}^{(k-1)} \cong \mathcal{E}^{(k)}, \text{ for } i \geq 1,$$

and an isomorphism

$$(4.1.2) \quad (\text{pr}_{01}^k)^*\phi: \partial_0^* \mathcal{E}^{(k-1)} \rightarrow \mathcal{E}^{(k)}, \text{ for } i = 0,$$

with the underlying isomorphisms of gerbes (3.3.1), (3.3.2) respectively.

The choice of the connection $\nabla^\mathcal{E}$ induces connections $\nabla_i^{\mathcal{E}^{(k)}}$ on $\mathcal{E}^{(k)}$, $i = 0, 1, \dots, k$ by the following formula

$$\nabla_i^{\mathcal{E}^{(k)}} = ((\text{pr}_{0i}^k)^*\phi^{-1})^*(\text{pr}_i^k)^*\nabla^\mathcal{E}$$

Finally, we pull back $\mathcal{E}^{(k)}$ to $\mathcal{G}^{(k)} \times \Delta^k$ via the projections on the first factor obtaining the bundle $\mathcal{E}_\Delta^{(k)}$, etc.

Note that we have a canonical isomorphism of twisted bundles

$$(4.1.3) \quad (\partial_i \times \text{id})^* \mathcal{E}_\Delta^{(k-1)} \cong (\text{id} \times \delta_i)^* \mathcal{E}_\Delta^{(k)}, \text{ for } i \geq 1$$

with the underlying isomorphism of gerbes (3.3.3).

For $i = 0$ we have an isomorphism of twisted bundles

$$(4.1.4) \quad (\text{pr}_{01}^k)^*\phi: (\partial_0 \times \text{id})^* \mathcal{E}_\Delta^{(k-1)} \longrightarrow (\text{id} \times \delta_0)^* \mathcal{E}_\Delta^{(k)}$$

with the underlying isomorphism of gerbes (3.3.4).

On the bundle $\mathcal{E}_\Delta^{(k)}$, we now consider the connection

$$(4.1.5) \quad (\nabla^\mathcal{E})^{(k)} = \sum_{i=0}^k t_i \nabla_i^{\mathcal{E}^{(k)}} + d_{dR}$$

where d_{dR} is the de Rham differential in t variables. Direct calculation shows the following:

Lemma 4.2. *Equip $\mathcal{E}_\Delta^{(k)}$ with the connection (4.1.5), and $(\text{pr}_{01}^k)^*\varsigma$ with the connection (3.3.7). Then (4.1.3) and (4.1.4) become isomorphisms of bundles with connections.*

Let $(\theta^\mathcal{E})^{(k)}$ be the curvature of $(\nabla^\mathcal{E})^{(k)}$. Use the isomorphisms (4.1.3) and (4.1.4) to identify $\text{End}((\partial_i \times \text{id})^* \mathcal{E}_\Delta^{(k-1)})$ with $\text{End}((\text{id} \times \delta_i)^* \mathcal{E}_\Delta^{(k)})$. Then Lemma 4.2 implies that we have

$$(\partial_i \times \text{id})^*(\theta^\mathcal{E})^{(k-1)} = (\text{id} \times \delta_i)^*(\theta^\mathcal{E})^{(k)}, \text{ for } i \geq 0.$$

Now given a \mathcal{G} -equivariant submersion $P \rightarrow \mathcal{G}^{(0)}$ as before, one can similarly define a notion of transversally twisted L -bundle. Recall the étale groupoid $\mathcal{P} = P \rtimes \mathcal{G}$ with source and range maps also denoted s and r respectively. The submersion $P \rightarrow \mathcal{G}^{(0)}$ gives a submersion $P \rtimes \mathcal{G} \rightarrow \mathcal{G}$ which is the pull-back under the range map and which is also a

groupoid morphism. Moreover, given a gerbe \mathcal{L} on $\mathcal{G}^{(0)}$ and a transversely \mathcal{L} -twisted bundle \mathcal{E} on $P \rightarrow \mathcal{G}^{(0)}$, the bundle $r^*\mathcal{E}$ is a transversally $r^*\mathcal{L}$ -twisted vector bundle over $P \rtimes \mathcal{G}$ with respect to the second projection $\hat{\pi} : P \rtimes \mathcal{G} \rightarrow \mathcal{G}$ and to the gerbe $r^*\mathcal{L}$ on \mathcal{G} .

Definition 4.3. An transversally L -twisted vector bundle is given by the following data:

- (1) An \mathcal{L} -transversely twisted bundle \mathcal{E} on $P \rightarrow \mathcal{G}^{(0)}$.
- (2) An isomorphism of transversally twisted vector bundles $\phi : s^*\mathcal{E} \rightarrow r^*\mathcal{E}$ over the submersion $P \rtimes \mathcal{G} \rightarrow \mathcal{G}$ such that:
 - The underlying isomorphism of gerbes is $\varsigma : s^*\mathcal{L} \rightarrow r^*\mathcal{L}$.
 - The 2-morphism $\nu : \varsigma_{01}^2 \circ \varsigma_{12}^2 \rightarrow \varsigma_{02}^2$ defines a 2-morphism $\nu : \phi_{01}^2 \circ \phi_{12}^2 \rightarrow \phi_{02}^2$ where $\phi_{ij}^2 := (\pi_{ij}^2)^*\phi$.

4.2. Projective \mathcal{G} -operators.

Assume as before that $\pi : P \rightarrow \mathcal{G}^{(0)}$ is a \mathcal{G} -equivariant submersion such that \mathcal{G} acts freely and properly on P , and let $L = (\mathcal{L}, \varsigma, \nu)$ be a \mathcal{G} -gerbe. We assume for simplicity, and in view of our interest in the index theorem, that the action of \mathcal{G} on P is cocompact. A \mathcal{G} -invariant distance on the \mathcal{G} -manifold P will be any smooth function $d = d_P$ on the closed submanifold $\{(p, p') \in P^2, \pi(p) = \pi(p')\}$ of P^2 such that

- (1) For any fiber $P_x := \pi^{-1}(x)$ of π , the restriction d_x of d to $P_x \times P_x$ is a distance which defines the induced topology of P_x .
- (2) For any $\gamma \in \mathcal{G}$ and any $(p, p') \in P_{r(\gamma)}^2$, we have $d_{s(\gamma)}(p\gamma, p'\gamma) = d_{r(\gamma)}(p, p')$.

It is easy to check using the properness of the action that such distance always exists. We fix such d and may use it in the definition of the proper support.

Let \mathcal{E} be a transversally L -twisted vector bundle on P . Recall the vector space $\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E})$ of properly supported \mathcal{L} -projective pseudo-differential operators on the submersion $P \rightarrow \mathcal{G}^{(0)}$. Then, $\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E})$ defines a \mathcal{G} -sheaf of algebras, and we describe the functorial isomorphisms now. First the similar space of (uniformly in \mathcal{G}) properly supported pseudodifferential operators, which are projective for the gerbe $r^*\mathcal{L}$, will suggestively be denoted $r^*\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E})$, so

$$r^*\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E}) := \Psi_{r^*\mathcal{L}}(P \rtimes \mathcal{G}|\mathcal{G}; r^*\mathcal{E}) \text{ with again } r(p, \gamma) = p.$$

We define similarly

$$s^*\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E}) := \Psi_{s^*\mathcal{L}}(P \rtimes \mathcal{G}|\mathcal{G}; s^*\mathcal{E}) \text{ and } s(p, \gamma) = p\gamma.$$

Notice that $s^*\mathcal{E}$ is now a transversally twisted vector bundle on $P \rtimes \mathcal{G}$ with respect to the gerbe $s^*\mathcal{L}$ on \mathcal{G} . The isomorphism $\varsigma : s^*\mathcal{L} \rightarrow r^*\mathcal{L}$ can then be pulled back to $P \rtimes \mathcal{G}$ using the second projection $\hat{\pi} : P \rtimes \mathcal{G} \rightarrow \mathcal{G}$ and yields an isomorphism of transversally twisted bundles

$$\hat{\pi}^*\varsigma : s^*\mathcal{E} \longrightarrow r^*\mathcal{E}.$$

We get in this way the functorial isomorphisms of sheaves of algebras by considering

$$\Psi(\varsigma) : s^*\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E}) \longrightarrow r^*\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E}),$$

see Lemma 2.33.

Given an operator $A \in \Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E})$, we define the operators s^*A and r^*A in $s^*\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E})$ and $r^*\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E})$ respectively, as follows. Recall that s and r are also local diffeomorphisms on $P \rtimes \mathcal{G}$ such that we may take as domains open sets of the form $\pi^{-1}U$ for $U \subset \mathcal{G}^{(0)}$. Therefore, we can use the following commutative diagrams and the usual pull-back of local vertical pseudodifferential operators under fiberwise preserving diffeomorphisms

$$\begin{array}{ccc} P \rtimes \mathcal{G} & \xrightarrow{\hat{r}} & P \\ \hat{\pi} \downarrow & & \downarrow \pi \\ \mathcal{G} & \xrightarrow{r} & \mathcal{G}^{(0)} \end{array} \quad \text{and} \quad \begin{array}{ccc} P \rtimes \mathcal{G} & \xrightarrow{\hat{s}} & P \\ \hat{\pi} \downarrow & & \downarrow \pi \\ \mathcal{G} & \xrightarrow{s} & \mathcal{G}^{(0)} \end{array}$$

Indeed, it is easy to check that we obtain in this way transversely twisted pseudodifferential operators r^*A and s^*A with respect to $r^*\mathcal{L}$ and $s^*\mathcal{L}$, which are, uniformly in \mathcal{G} , properly supported.

Definition 4.4. The vector space of \mathcal{G} -invariant (fiberwise) pseudo-differential operators $\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E})^{\mathcal{G}}$ is defined to be $\{A \in \Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E}) \mid \Psi(\varsigma)(s^*A) = r^*A\}$.

We pause here to give an explicit example.

4.3. An example of twisted \mathcal{G} -family.

Let B be a smooth manifold on which the group \mathbb{Z} acts. We denote the (action of) generator of \mathbb{Z} by T . Let \mathcal{G} be the corresponding action groupoid: $\mathcal{G}^{(0)} = B$, $\mathcal{G}^{(1)} = B \times \mathbb{Z}$ with $s(b, n) = b$, $r(b, n) = T^n b$, $(b, n) \in B \times \mathbb{Z}$. Let ℓ be a \mathbb{Z} equivariant line bundle on B . We can construct a \mathcal{G} -gerbe L as follows.

- The gerbe \mathcal{L} on the unit space $\mathcal{G}^{(0)}$ is trivial.
- The isomorphism $\varsigma: s^*\mathcal{L} \rightarrow r^*\mathcal{L}$ of gerbes on $\mathcal{G}^{(1)}$ is given by $\varsigma|_{B \times n} = \ell^{\otimes n}$, $n \in \mathbb{Z}$.
- A 2-morphism $\nu: \varsigma_{01}^2 \circ \varsigma_{12}^2 \rightarrow \varsigma_{02}^2$ is given by the isomorphism $(T^n)^*\ell^{\otimes m} \otimes \ell^{\otimes n} \rightarrow \ell^{\otimes(m+n)}$ on $B \times m \times n$, induced by the isomorphism $(T^n)^*\ell^{\otimes m} \rightarrow \ell^{\otimes m}$.

Note that we have canonical isomorphism $H^\bullet(\mathcal{G}; \mathbb{C}) \cong H^\bullet(B_T; \mathbb{C})$ where B_T is the mapping torus of the map $T: B_T := (B \times [0, 1]) / \sim$. Here \sim is the equivalence relation given by $(b, 1) \sim (Tb, 0)$. Under this identification the de Rham class of L can be described by the following 3-form $\Omega \in \Omega^3(B_T)$. Let $\phi \in C_c^\infty(0, 1)$ be such that $\int_0^1 \phi(t) dt = 1$. Chose a connection ∇ on ℓ and let $\theta \in \Omega^2(B)$ be its curvature. Then the form $\theta\phi(t)dt \in \Omega^3(B \times [0, 1])$ descends to a well defined closed 3 form Ω on B_T . It is easy to see that the cohomology class of this form is, in general, nontrivial.

Let \widehat{B} be the circle bundle of ℓ , $\widehat{B} = \{\ell \setminus 0\} / \mathbb{R}^{>0}$, and let $p: \widehat{B} \rightarrow B$ be the natural projection. The \mathbb{Z} action on B naturally induces a \mathbb{Z} action on \widehat{B} , since ℓ is equivariant. Choose an Hermitian metric on ℓ . This choice induces canonically a trivialization of $p^*\ell$, and let $s \in \Gamma(\widehat{B}; p^*\ell)$ be the corresponding section. If we define the form $\alpha \in \Omega^1(\widehat{B})$ (depending on s) by $(p^*\nabla)(s) = \alpha s$, then

$$d\alpha = p^*\theta.$$

It follows that with $\eta = \alpha\phi(t)dt \in \Omega^2(\widehat{B}_T)$ we have $d\eta = p^*\Omega$. Here \widehat{B}_T is the mapping torus of \mathbb{Z} action on \widehat{B} . It follows that the map

$$I_\eta: \xi \mapsto e^{-u\eta}\xi$$

intertwines de Rham differential ud with d_Ω and induces an isomorphism between the twisted and untwisted cohomology of \widehat{B}_T .

Assume that M is a compact spin manifold and $\widetilde{M} \rightarrow M$ is a \mathbb{Z} -covering. Let ω be an S^1 -equivariant line bundle on $M \times S^1$. denote by D a $\mathbb{Z} \times S^1$ equivariant Dirac-type operator on $\widetilde{M} \times S^1$, acting on sections of the spinor bundle twisted by ω . Set $P := \widetilde{M} \times \widehat{B}$; then $\pi: P \rightarrow B$ given by $\pi(m, b) = p(b)$, $m \in \widetilde{M}, b \in \widehat{B}$, is naturally a \mathcal{G} -fibration. We will denote by \mathcal{E} for the pull back of the spinor bundle tensored with ω to P . It is a \mathbb{Z} equivariant bundle on P . It can be endowed with a structure of L - transversally twisted bundle. To do this we need to specify isomorphism of twisted vector bundles $\phi: s^*\mathcal{E} \rightarrow r^*\mathcal{E}$. In this example is given by isomorphism

$$\pi^*\ell^{\otimes n} \otimes \mathcal{E} \rightarrow (T^n)^*\mathcal{E}$$

given by the trivialization of $\pi^*\ell$ (induced by the trivialization of $p^*\ell$) and the isomorphism $(T^n)^*\mathcal{E} \cong \mathcal{E}$ (given by \mathbb{Z} -equivariance of \mathcal{E}). The Dirac operator D induces a constant family on the fibration $(\widetilde{M} \times S^1) \times \widehat{B} \rightarrow \widehat{B}$ which descends, since D is S^1 -equivariant, to the family on $P \rightarrow B$ acting on the sections of \mathcal{E} . It is easily seen to be \mathcal{G} -invariant.

4.4. Invariant operators and cross-products.

For any c-soft \mathcal{G} sheaf of algebras \mathcal{A} , one can define the convolution algebra $\mathcal{A} \rtimes \mathcal{G}$ as follows, see [28], 3.31 for the details. As a vector space it is the space of compactly supported sections $\Gamma_c(\mathcal{G}; \mathcal{A})$. The convolution product is given by

$$fg(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1)(\gamma_1 \cdot g(\gamma_2)).$$

Now the sheaf of algebras $\Psi_{\mathcal{L}}(\pi; \mathcal{E})$ associated with $\pi: P \rightarrow \mathcal{G}^{(0)}$ and defined in Remark 2.32 being a c-soft \mathcal{G} -sheaf, we eventually obtain the well-defined cross-product algebra that we denote for simplicity by $\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E}) \rtimes \mathcal{G}$. As a space, $\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E}) \rtimes \mathcal{G}$ can be identified with the space of properly supported operators from $\Psi_{r^*\mathcal{L}}(P \rtimes \mathcal{G}|\mathcal{G}; r^*\mathcal{E})$ such that the projection of the support in \mathcal{G} is compact, and with the convolution product defined above.

Consider now $\phi \in C_c^\infty(P; \mathbb{R})$ such that for any $p \in P$,

$$\sum_{\{\gamma | r(\gamma) = \pi(p)\}} \phi^2(p\gamma) = 1.$$

Let $\gamma \in \mathcal{G}$. Define $\phi^\gamma \in C^\infty(\pi^{-1}(r(\gamma)))$ by $\phi^\gamma(p) := \phi(p\gamma), p \in r(\gamma)$. Notice that ϕ^γ is a restriction to $\pi^{-1}(r(\gamma))$ of a smooth function defined on $\pi^{-1}(U)$ for a sufficiently small neighborhood U of $r(\gamma)$. Indeed, let V be a neighborhood of γ in \mathcal{G} such that r and s restricted to V are diffeomorphisms. Then we can take $U = r(V)$. Notice that the action of $\gamma: \pi^{-1}(r(\gamma)) \rightarrow \pi^{-1}(s(\gamma))$ extends to a diffeomorphism, which we also denote by $\gamma: \pi^{-1}(r(V)) \rightarrow \pi^{-1}(s(V))$, and this defines the extension of ϕ^γ to $\pi^{-1}(U)$. Note also that $\phi^\gamma(p)$ is smooth in $(\gamma, p) \in V \times \pi^{-1}(V)$, $\pi(p) = r(\gamma)$. The straightforward verification of the following lemma is left to the reader.

Lemma 4.5. *The map*

$$I: \Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E})^{\mathcal{G}} \rightarrow \Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \mathcal{E}) \rtimes \mathcal{G} \text{ defined by } I(A)(\gamma) = \phi A \phi^\gamma,$$

is an algebra homomorphism.

4.5. Analytic index of projective \mathcal{G} -operators.

Definition 4.6. Let $m \geq 0$ be fixed. A fiberwise pseudodifferential operator

$$A \in \Psi_{\mathcal{L}}^m(P|\mathcal{G}^{(0)}; \mathcal{E}, \mathcal{E}')$$

is fiberwise elliptic if the principal symbol $\sigma_m(A) \in \Gamma(p^* \text{Hom}(\mathcal{E}, \mathcal{E}'))$ is an isomorphism.

We say that $Q \in \Psi_{\mathcal{L}}^{-m}(P|\mathcal{G}^{(0)}; \mathcal{E}', \mathcal{E})$ is a parametrix of $A \in \Psi_{\mathcal{L}}^m(P|\mathcal{G}^{(0)}; \mathcal{E}, \mathcal{E}')$ if $A \circ Q - 1 \in \Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}; \mathcal{E}', \mathcal{E}')$ and $Q \circ A - 1 \in \Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}; \mathcal{E}, \mathcal{E})$. Recall that the action of \mathcal{G} on P is free proper and co-compact.

Lemma 4.7. Every \mathcal{G} -invariant elliptic operator $A \in \Psi_{\mathcal{L}}^m(P|\mathcal{G}^{(0)}; \mathcal{E}, \mathcal{E}')^{\mathcal{G}}$ has a \mathcal{G} -invariant parametrix Q in $\Psi_{\mathcal{L}}^{-m}(P|\mathcal{G}^{(0)}; \mathcal{E}', \mathcal{E})^{\mathcal{G}}$.

Proof. Consider the algebra of \mathcal{G} -invariant transversally L -twisted fiberwise pseudo-differential operators $\Psi_{\mathcal{L}}^m(P|\mathcal{G}^{(0)}; \mathcal{E} \oplus \mathcal{E}', \mathcal{E} \oplus \mathcal{E}')^{\mathcal{G}}$, and the element

$$\mathcal{A} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}.$$

where A^* is a formal adjoint of A , an element of $\Psi_{\mathcal{L}}^m(P|\mathcal{G}^{(0)}; \mathcal{E}', \mathcal{E})^{\mathcal{G}}$, defined using some unitary structures on \mathcal{L} and $\mathcal{E} \oplus \mathcal{E}'$.

Consider a \mathcal{G} -invariant distance d on P as before. There exists a \mathcal{G} -invariant function $k : P \times P \rightarrow \mathbb{R}$ and $a, b \in \mathbb{R}^{+*}$ such that $k(x, y) = 1$ for $d(x, y) < a$ and $k(x, y) = 0$ for $d(x, y) > b$. Consider the function of the real variable $f(u) = \frac{1-e^{-u^2}}{u}$. One then checks that the operator with Schwartz kernel $k(x, y)K_{f(\mathcal{A})}(x, y)$ defines a \mathcal{G} -invariant parametrix for \mathcal{A} . One then immediately obtains a \mathcal{G} -invariant parametrix (uniformly properly supported) for A . \square

Let $D^+ \in \Psi_{\mathcal{L}}^m(P|\mathcal{G}^{(0)}; \mathcal{E}, \mathcal{E}')^{\mathcal{G}}$ be elliptic and denote by $S^*(P|\mathcal{G}^{(0)})$ the vertical cosphere bundle of the fibration π . Let $F \in \Psi_{\mathcal{L}}^0(P|\mathcal{G}^{(0)}; \mathcal{E}, \mathcal{E}')^{\mathcal{G}}$ be such that

$$\sigma_0(F)|_{S^*(P|\mathcal{G}^{(0)})} = \sigma_m(D^+)|_{S^*(P|\mathcal{G}^{(0)})}.$$

Choose a parametrix G for F . Let $U_{D^+} \in \Psi_{\mathcal{L}}^0(P|\mathcal{G}^{(0)}; \mathcal{E} \oplus \mathcal{E}')^{\mathcal{G}}$ be an invertible operator such that $U_{D^+} - \begin{bmatrix} 0 & -G \\ F & 0 \end{bmatrix} \in \Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}; \mathcal{E} \oplus \mathcal{E}')^{\mathcal{G}}$. An explicit construction of an example of such

operator is as follows. Let $S_0 = 1 - GF$, $S_1 = 1 - FG$. Then set $U_{D^+} = \begin{bmatrix} S_0 & -(1 + S_0)G \\ F & S_1 \end{bmatrix}$.

With such a choice the inverse is given by an explicit formula $U_{D^+}^{-1} = \begin{bmatrix} S_0 & (1 + S_0)G \\ -F & S_1 \end{bmatrix}$.

Definition 4.8. The index of D^+ is the class in the K -theory of the algebra $\Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}; \mathcal{E} \oplus \mathcal{E}')^{\mathcal{G}}$ defined by

$$\begin{aligned} \text{ind}(D^+) &= [P_{D^+} - Q] \in K_0(\Psi_{\mathcal{L}}^0(P|\mathcal{G}^{(0)}, \mathcal{E} \oplus \mathcal{E}')^{\mathcal{G}}, \Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E} \oplus \mathcal{E}')^{\mathcal{G}}) \\ &\cong K_0(\Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E} \oplus \mathcal{E}')^{\mathcal{G}}), \end{aligned}$$

where P_{D^+} and Q are the idempotents given by $P_{D^+} = U_{D^+} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U_{D^+}^{-1}$ and $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Remark 4.9. *One can use the regular representation to complete the, twisted by L , convolution algebra of \mathcal{G} to a C^* -algebra $C_r^*(\mathcal{G}, L)$ and define the analytic index in its K -theory.*

We leave to the reader the standard K -theoretic proof that the index is well defined and is stable under the homotopies of D^+ in the class of elliptic operators in $\Psi_{\mathcal{L}}^m(P|\mathcal{G}^{(0)}; \mathcal{E}, \mathcal{E}')^{\mathcal{G}}$. Assume that the transversally \mathcal{L} -twisted bundles \mathcal{E} and \mathcal{E}' are hermitian (and in particular \mathcal{L} is unitary) and that the fibers of π are equipped with \mathcal{G} -equivariant smoothly varying volume forms. In this situation for a projective family D^+ we can again define a formally adjoint projective family $(D^+)^*$ by forming the formal adjoints for each family D_{α} .

Lemma 4.10. *Then, identifying $K_0(\Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E} \oplus \mathcal{E}')^{\mathcal{G}})$ with $K_0(\Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}' \oplus \mathcal{E})^{\mathcal{G}})$ we have*

$$\text{ind}(D^+)^* = -\text{ind } D^+.$$

Proof. We have

$$\text{ind}(D^+)^* = \left[U_{(D^+)^*} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} U_{(D^+)^*}^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right] \in K_0(\Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E} \oplus \mathcal{E}')^{\mathcal{G}}).$$

By deforming in the usual way D^+ we may assume that $\sigma_m(D^+)|_{S^*(P|\mathcal{G}^{(0)})}$ is an isometry. In this case we may choose $U_{(D^+)^*} = U_{D^+}^{-1}$, and the statement follows. \square

5. THE TWISTED CONNES MAP

5.1. Review of cyclic homology. Let us first review some basic material on the cyclic theory. The general reference is the book [42], here we review the basic definitions to fix the notations.

Let A be a complex unitary algebra. Set $C_k(A) = A \otimes (A/\mathbb{C}1)^{\otimes k}$ and $C^k(A)$ for the dual vector space of $C_k(A)$. Let u be a formal variable of degree -2 . The space of negative cyclic chains of degree $l \in \mathbb{Z}$ is defined by

$$\text{CC}_l^-(A) = (C_{\bullet}(A)[[u]])_l = \prod_{-2n+k=l, n \geq 0} u^n C_k(A).$$

The boundary is given by $b+uB$ where b and B are the Hochschild and Connes boundaries of the cyclic complex. The homology of this complex is denoted $\text{HC}_{\bullet}^-(A)$. When the algebra A is \mathbb{Z}_2 graded they incorporate the relevant signs. If A is not necessarily unital denote by A^+ its unitalisation and set $\text{CC}_l^-(A) = \text{CC}_l^-(A^+)/\text{CC}_l^-(\mathbb{C})$. If I is an ideal in a unital algebra A the relative cyclic complex is defined by $\text{CC}_{\bullet}^-(A, I) = \text{Ker}(\text{CC}_{\bullet}^-(A) \rightarrow \text{CC}_{\bullet}^-(A/I))$. One has a natural morphism of complexes $\iota: \text{CC}_{\bullet}^-(I) \rightarrow \text{CC}_{\bullet}^-(A, I)$ induced by the homomorphism $I^+ \rightarrow A$.

Dually, we define the cyclic cohomology complex by

$$\text{CC}^l(A) = (C^{\bullet}(A)[[u^*]])_l = \bigoplus_{2n+k=l, n \geq 0} (u^*)^n C^k(A).$$

with coboundary given by $b + u^*B$, where u^* has degree $+2$.

Recall that for an algebra A we have the Chern character in cyclic homology $\text{ch}: K_0(A) \rightarrow HC_0^-(A)$. It is defined by the following formula. Let $P, Q \in M_n(A^+)$ be two idempotents in $n \times n$ matrices of the algebra A^+ , representing a class $[P - Q] \in K_0(A)$. Then

$$(5.1.1) \quad \text{Ch}([P - Q]) = \text{tr}(P - Q) + \sum_{n=1}^{\infty} (-u)^n \frac{(2n)!}{n!} \text{tr} \left(\left(P - \frac{1}{2} \right) \otimes P^{\otimes(2n)} - \left(Q - \frac{1}{2} \right) \otimes Q^{\otimes(2n)} \right)$$

We will use abusively the notation $\text{Ch}([P - Q])$ for the cyclic cycle defined above and $\text{ch}([P - Q])$ for its well defined class in cyclic homology $HC_0^-(A)$.

We will also need to use the entire cyclic complex. For our purpose the algebraic version from [25], IV.7.α Remark 7 b. will be sufficient. First recall that one has the periodic cyclic complex $(CC_{\bullet}^{\text{per}}(A), b + uB)$ where $CC_{\bullet}^{\text{per}}(A) = C_{\bullet}(A)[u^{-1}, u]$. Assume we are given a periodic chain $\alpha = \sum_{k \geq 0} \alpha_k u^k \in CC_m^{\text{per}}(A)$, $\alpha_k \in C_{2k+m}(A)$. Then α is called entire if there exist a finite dimensional subspace $V \subset A$, $1 \in V$ and $C > 0$ (depending on α) such that $\alpha_k \in V \otimes (V/\mathbb{C}1)^{\otimes k}$ and $\|\alpha_k\| \leq C^k k!$. Here the norms on $V \otimes (V/\mathbb{C}1)^{\otimes k}$ are induced by an arbitrary norm on V . We denote the entire cyclic complex of A by $CC_{\bullet}^{\text{entire}}(A)$.

Note that the chain $\text{Ch}([P - Q])$ defined in (5.1.1) is an element in $CC_0^{\text{entire}}(A)$.

5.2. The Connes map.

Assume first that $M \rightarrow B$ is a submersion. Let \mathcal{L} be a gerbe with connection on B and \mathcal{E} a transversally \mathcal{L} -twisted bundle on M with connection. Let $\Omega(\mathcal{L})$ be the associated curvature.

Following Mathai and Stevenson [47], we construct a morphism of complexes

$$\Phi_{\nabla^{\mathcal{H}}} : \mathcal{C}_{\Omega(\mathcal{L})}^{\bullet}(B) \longrightarrow CC^{\bullet}(\Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E}))$$

as follows. Denote by Δ^l the standard l -simplex

$$\Delta^l := \{(\sigma_0, \dots, \sigma_l) \in \mathbb{R}^{l+1} \mid 0 \leq \sigma_i, \sum_{i=0}^l \sigma_i = 1\}.$$

Set for any $\xi \in \mathcal{C}_{\bullet}(B)[u^*]$,

$$\Phi_{\nabla^{\mathcal{H}}}(\xi) = (-1)^{N(\xi)} \sum_k (u^*)^k \phi_k(\xi),$$

where $N(\xi) = \frac{\deg \xi (\deg \xi + 1)}{2}$ and the cochain $\phi_k(\xi) \in C^l(\Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E}))$ for $l = \deg \xi - 2k$, is defined by:

$$(5.2.1) \quad \phi_k(\xi)(A_0, \dots, A_l) := \int_{\Delta^l} \left\langle (u^*)^{-k} \xi, \text{Tr} \left(A_0 e^{-\sigma_0 u \theta^{\mathcal{H}}} \partial^{\mathcal{H}}(A_1) e^{-\sigma_1 u \theta^{\mathcal{H}}} \dots e^{-\sigma_{l-1} u \theta^{\mathcal{H}}} \partial^{\mathcal{H}}(A_l) e^{-\sigma_l u \theta^{\mathcal{H}}} \right) \right\rangle d\sigma_1 \dots d\sigma_l$$

where $A_1, \dots, A_l \in \Psi_{\mathcal{L}}^{-\infty}$, $A_0 \in \Psi_{\mathcal{L}}^{-\infty} + \mathbb{C} \cdot 1$, with the convention that $\text{Tr}(1) = 0$. Recall the twisted de Rham complexes reviewed in Appendix A.

Theorem 5.1 (cf. [47]). *The map*

$$(5.2.2) \quad \Phi_{\nabla^{\mathcal{H}}} : \mathcal{C}_{\Omega(\mathcal{L})}^{\bullet}(B) \longrightarrow \mathrm{CC}^{\bullet}(\Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E}))$$

defined above is a morphism of complexes.

Proof. This is an immediate consequence of the result of [47][page 323-] and an observation that our map is the transposed of the map they construct. \square

This morphism depends on the choice of our horizontal distribution \mathcal{H} . However the results of [47] show that a different choice of \mathcal{H} leads to a chain homotopic morphism.

Remark 5.2. *Note that the similar morphism in the non-twisted case was constructed in the references [5, 35, 51, 52].*

We now proceed to explain the construction of the Φ -map in our general case. Our constructions are based on the previous results obtained for discrete translation groupoids in [1].

Assume again that we have an étale groupoid \mathcal{G} , and a gerbe L with connection on \mathcal{G} . Assume that $\pi : P \rightarrow \mathcal{G}^{(0)}$ is a \mathcal{G} -equivariant submersion and that \mathcal{E} is a transversally L -twisted vector bundle with connection on P . Assume as before that the action of \mathcal{G} on P is free, proper and for simplicity cocompact. Recall that we also have for every $k \in \mathbb{N}$, a submersion $\mathcal{P}^{(k)} \rightarrow \mathcal{G}^{(k)}$, a gerbe $\mathcal{L}^{(k)}$ on $\mathcal{G}^{(k)}$ and a transversally $\mathcal{L}^{(k)}$ -twisted vector bundle $\mathcal{E}^{(k)}$ on $\mathcal{P}^{(k)}$ (recall that $\mathcal{P}^{(k)}$ is defined in Section 1). We thus obtain the following equality of sheaves

$$\Psi_{\mathcal{L}^{(k)}}(\pi^{(k)}; \mathcal{E}^{(k)}) = (\mathrm{pr}_0^k)^* \Psi_{\mathcal{L}}(\pi; \mathcal{E})$$

on every $\mathcal{G}^{(k)}$. See Subsection 2.3 for the definitions. Using the constructions of Section 4 we obtain for every k a submersion $\mathcal{P}^{(k)} \times \Delta^k \rightarrow \mathcal{G}^{(k)} \times \Delta^k$, a gerbe $\mathcal{L}_{\Delta}^{(k)}$ on $\mathcal{G}^{(k)} \times \Delta^k$ and a transversally $\mathcal{L}_{\Delta}^{(k)}$ -twisted bundle $\mathcal{E}_{\Delta}^{(k)}$ on $\mathcal{P}^{(k)} \times \Delta^k$.

We can view the algebra of operators $\Psi_{\mathcal{L}^{(k)}}(\mathcal{P}^{(k)}|\mathcal{G}^{(k)}; \mathcal{E}^{(k)})$ as a subalgebra in $\Psi_{\mathcal{L}_{\Delta}^{(k)}}(\mathcal{P}^{(k)} \times \Delta^k|\mathcal{G}^{(k)} \times \Delta^k; \mathcal{E}_{\Delta}^{(k)})$ constant in the Δ^k direction, i.e. pulled back to $\mathcal{G}^{(k)} \times \Delta^k$ via the projection on the first factor.

Recall that the gerbe $\mathcal{L}_{\Delta}^{(k)}$ is equipped with connection given by (3.3.5), (3.3.6) and $\mathcal{E}_{\Delta}^{(k)}$ is equipped with connection given in (4.1.5). Since the action of \mathcal{G} on P is proper, there exists a horizontal distribution \mathcal{H} on $P \rightarrow \mathcal{G}^{(0)}$ which is \mathcal{G} -invariant. We therefore obtain a derivation (cf. (2.4.2))

$$\partial^{(k)} : \Psi_{\mathcal{L}_{\Delta}^{(k)}}(\mathcal{P}^{(k)} \times \Delta^k|\mathcal{G}^{(k)} \times \Delta^k; \mathcal{E}_{\Delta}^{(k)}) \longrightarrow \Omega^1(\mathcal{G}^{(k)}, \Psi_{\mathcal{L}_{\Delta}^{(k)}}(\mathcal{P}^{(k)} \times \Delta^k|\mathcal{G}^{(k)} \times \Delta^k; \mathcal{E}_{\Delta}^{(k)})),$$

and we get by (2.4.3), $\theta^{(k)} \in \Omega^2(\mathcal{G}^{(k)} \times \Delta^k, \Psi_{\mathcal{L}_{\Delta}^{(k)}}(\mathcal{P}^{(k)} \times \Delta^k|\mathcal{G}^{(k)} \times \Delta^k; \mathcal{E}_{\Delta}^{(k)}))$, such that $(\partial^{(k)})^2 = [\theta^{(k)}, \cdot]$. We then have by straightforward verification the following identities

$$(\partial_i \times \mathrm{id})^* \partial^{(k-1)} = (\mathrm{id} \times \delta_i)^* \partial^{(k)} \text{ for } i \geq 0.$$

$$(\partial_i \times \mathrm{id})^* \theta^{(k-1)} = (\mathrm{id} \times \delta_i)^* \theta^{(k)} \text{ for } i \geq 0.$$

Note also that $\theta^{(k)}$ has components of $(2, 0)$ and $(1, 1)$ types only.

For a cocycle $c = \{c_k\} \in \mathcal{C}_\Delta(\mathcal{G})^\bullet$, we set

$$\Phi^\mathcal{E}(c) = (-1)^{N(c)} \sum (u^*)^j \phi_j(c)$$

where $\phi_j(c) = \sum_{k+l=\deg c-2j} \phi_j^{kl}(c_k)$ and $\phi_j^{kl}(c_k) \in \text{CC}^\bullet(\Psi_\mathcal{L}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}))$ is given by

$$(5.2.3) \quad \phi_j^{kl}(c_k)(A_0, \dots, A_l) := \int_{\Delta^l} \left\langle (u^*)^{-j} c_k, \text{Tr} \left(A_0 e^{-\sigma_0 u \theta^{(k)}} \partial^{(k)}(A_1) e^{-\sigma_1 u \theta^{(k)}} \dots e^{-\sigma_{l-1} u \theta^{(k)}} \partial^{(k)}(A_l) e^{-\sigma_l u \theta^{(k)}} \right) \right\rangle d\sigma_1 \dots d\sigma_l$$

Note that this expression is 0 if $l > \dim \mathcal{G}^{(0)}$ or $k > \dim \mathcal{G}^{(0)} + \deg \omega$, and therefore the sum in the formula for $\Phi^\mathcal{E}$ is finite.

Recall that the complex $\mathcal{C}_\Delta(\mathcal{G})^\bullet$ is endowed with the differential $d_{\Omega(L)}^t$ defined by (1.2.1). We then have the following simplicial analogue of Theorem 5.1

Theorem 5.3. *The map $\Phi^\mathcal{E}$ is a morphism of complexes*

$$\Phi^\mathcal{E}: (\mathcal{C}_\Delta(\mathcal{G})^\bullet, d_{\Omega(L)}^t) \longrightarrow C^\bullet(\mathcal{G}, \text{CC}^\bullet(\Psi_\mathcal{L}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}))).$$

Recall that $\text{CC}^\bullet(\Psi_\mathcal{L}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}))$ defines a \mathcal{G} -sheaf of algebras and the complex on the right hand side of the theorem is thus well defined, see also Appendix B. The proof is an immediate rephrasing of the classical one of Theorem 5.1 and is omitted. The above theorem is also an extension of the similar statement for discrete translation groupoids proved in [1] and which corresponds to suspended foliations.

Using again the notations from Appendix B and composing the induced morphism on cohomology with the map

$$H^\bullet(C^\bullet(\mathcal{G}, \text{CC}^\bullet(\Psi_\mathcal{L}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E})))) \longrightarrow \text{HC}^\bullet(\Psi_\mathcal{L}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}) \rtimes \mathcal{G}),$$

which comes from [15, 28], and then with the morphism

$$I^*: \text{HC}^\bullet(\Psi_\mathcal{L}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}) \rtimes \mathcal{G}) \longrightarrow \text{HC}^\bullet(\Psi_\mathcal{L}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E})^\mathcal{G})$$

induced by the map I constructed in Lemma 4.5, we eventually end up with the map

$$\Lambda^\mathcal{E}: H^\bullet(\mathcal{C}_\Delta(\mathcal{G})^\bullet, d_{\Omega(L)}^t) \longrightarrow \text{HC}^\bullet(\Psi_\mathcal{L}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E})^\mathcal{G}).$$

Said differently, we end up with the morphism

$$(5.2.4) \quad \Lambda^\mathcal{E}: H_{\Omega(L)}^\bullet(B\mathcal{G}) \longrightarrow \text{HC}^\bullet(\Psi_\mathcal{L}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E})^\mathcal{G}).$$

Assume now that the bundle \mathcal{E} is \mathbb{Z}_2 graded, $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ with the grading operator γ , $\gamma|_{\mathcal{E}^\pm} = \pm 1$. In this case $\Psi_\mathcal{L}^{-\infty}(\pi, \mathcal{E})$ is a sheaf of \mathbb{Z}_2 graded algebras with the even part $(\Psi_\mathcal{L}^{-\infty}(\pi, \mathcal{E}))^+ = \{A|A\gamma = \gamma A\}$.

We can use the supertrace $\text{Str} = \text{Tr} \circ \gamma$ instead of the trace in the formula (5.2.3) to obtain cyclic cochains on the algebras $\Psi_{\mathcal{L}^{(k)}}^{-\infty}(\mathcal{P}^{(k)}|\mathcal{G}^{(k)}; \mathcal{E}^{(k)})^+$ and the following generalization of Theorem 5.3.

Theorem 5.4. *We have a morphism of complexes*

$$\Phi^{\mathcal{E}, \gamma}: (\mathcal{C}_\Delta(\mathcal{G})^\bullet, d_{\Omega(L)}^t) \longrightarrow C^\bullet(\mathcal{G}, \text{CC}^\bullet(\Psi_\mathcal{L}^{-\infty}((P|\mathcal{G}^{(0)}, \mathcal{E}))^+)),$$

which allows to define our twisted map on cohomology:

$$\Lambda^{\mathcal{E}, \gamma}: H_{\Omega(L)}^{\bullet}(B\mathcal{G}) \longrightarrow \mathrm{HC}^{\bullet}((\Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}))^{+})^{\mathcal{G}}).$$

We are now in position to define the higher c -index associated with any element $[c] \in H^{\bullet}(\mathcal{C}_{\Delta}(\mathcal{G})^{\bullet}, d_{\Omega(L)}^t)$. More precisely,

Definition 5.5. We define the index pairing for a \mathcal{G} -invariant L -projective elliptic family D , with the groupoid cocycle c as

$$\mathrm{ind}_c(D) := \langle (u^*)^{-\frac{\deg c}{2}} \Lambda^{\mathcal{E}}(c), \mathrm{ch}(\mathrm{ind} D) \rangle.$$

This higher twisted c -index $\mathrm{ind}_c(D)$ is a complex number which is computed by a local formula *à la Atiyah-Singer* in the next section.

6. THE HIGHER TWISTED INDEX FORMULA

The goal of this section is to give a superconnection construction of Connes' Φ map which will be used to prove our higher index theorem for L -projective leafwise Dirac operators. Recall that we assume that \mathcal{G} acts freely, properly and co-compactly on P .

6.1. Projective superconnections on fibrations. We first review some material from [4] and consider the smooth submersion $\pi: M \rightarrow B$. Assume that the fibers of π are even dimensional complete Riemannian manifolds. Denote by $C(M|B)$ the Clifford algebra of the fiberwise cotangent bundle $T^*(M|B) = T^*M/(\ker \pi_*)^{\perp}$. Let \mathcal{L} be a unitary gerbe on B .

Definition 6.1.

- A twisted Clifford module is a transversally \mathcal{L} -twisted Hermitian \mathbb{Z}_2 -graded vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ on M together with the homomorphism $c: C(M|B) \rightarrow \mathrm{End}(\mathcal{E})$ of bundles of unital \mathbb{Z}_2 -graded $*$ -algebras.
- A Clifford connection $\nabla^{\mathcal{E}}$ on \mathcal{E} is an Hermitian connection such that for any section s of \mathcal{E} one has:

$$\nabla^{\mathcal{E}}(c(a)s) = c(\nabla^{M|B}(a))s + c(a)\nabla^{\mathcal{E}}(s).$$

Clifford connections on Clifford modules always exist; the proof is analogous to the proof of existence of twisted connections in Lemma 2.20.

Choose a horizontal distribution $\mathcal{H} \subset TM$ so that $TM = \mathcal{H} \oplus T(M|B)$. This choice together with the Riemannian metric on the fibers of π allows one to define a connection $\nabla^{M|B}$ on the fiberwise tangent bundle $T(M|B)$, see [7] Section 10.1. We denote by $R^{M|B}$ the curvature of this connection. Set

$$\mathrm{End}_{C(M|B)}(\mathcal{E}) = \{A \in \mathrm{End}(\mathcal{E}) \mid [A, c(a)] = 0 \text{ for every } a \in C(M|B)\}.$$

Let $\Gamma \in C(M|B)$ be the chirality operator defined locally by $\Gamma = i^{k/2}e^1 \dots e^k$ where $k = \dim M - \dim B$ and e^1, \dots, e^k is the local orthonormal basis of $T^*(M|B)$. Define then the relative supertrace

$$\mathrm{str}_{\mathcal{E}/S}: \mathrm{End}_{C(M|B)}(\mathcal{E}) \longrightarrow C^{\infty}(M) \text{ by } \mathrm{str}_{\mathcal{E}/S}(A) = 2^{-k/2} \mathrm{str} c(\Gamma)A.$$

We fix from now on a Clifford connection $\nabla^\mathcal{E}$ on \mathcal{E} and a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ for the transversally \mathcal{L} -twisted Clifford module \mathcal{E} . The connection $\nabla^\mathcal{E}$ defines a connection $(\nabla_\alpha^\mathcal{E}, \nabla_{\alpha\beta}, \omega_\alpha)$ (defined up to equivalence) on this descent datum. Each \mathcal{E}_α is then a Clifford module on the submersion $\pi^{-1}U_\alpha \rightarrow U_\alpha$, and each connection $\nabla_\alpha^\mathcal{E}$ is a Clifford connection.

Recall (see Proposition 2.22) that one defines $\theta^\mathcal{E} \in \Omega^2(M, \text{End}(\mathcal{E}))$ by setting $\theta^\mathcal{E}|_{\pi^{-1}U_\alpha} = \theta_\alpha^\mathcal{E} + \pi^*\omega_\alpha$. Denote by $c(R^{M|B})$ the action of the 2-form with values in the Clifford algebra obtained from $R^{M|B}$ via the Lie algebra isomorphism $\mathfrak{so}(T(M|B)) \rightarrow C^2(M|B)$. Here $C^2(M|B) \subset C(M|B)$ is the subspace consisting of elements $\sum u_i v_i$, $u_i, v_i \in T^*(M|B)$ with $\sum \langle u_i, v_i \rangle = 0$. Define $\theta^{\mathcal{E}/\mathcal{S}} = \theta^\mathcal{E} - c(R^{M|B})$.

The argument in [7], Proposition 3.43, shows that $\theta^{\mathcal{E}/\mathcal{S}} \in \Omega^2(M, \text{End}_{C(M|B)}(\mathcal{E}))$. Over an open set $\pi^{-1}U_\alpha$ we have $\theta^{\mathcal{E}/\mathcal{S}}|_{U_\alpha} = \theta_\alpha^{\mathcal{E}/\mathcal{S}} + \pi^*\omega_\alpha$ where $\theta_\alpha^{\mathcal{E}/\mathcal{S}} \in \text{End}_{C(M|B)}(\mathcal{E}_\alpha)$ is defined via the equality $\theta_\alpha^\mathcal{E} = \theta_\alpha^{\mathcal{E}/\mathcal{S}} + c(R^{M|B})$. We can then define a differential form $\text{Ch}_\mathcal{L}(\mathcal{E}/\mathcal{S})$ by

$$(6.1.1) \quad \text{Ch}_\mathcal{L}(\mathcal{E}/\mathcal{S}) := \text{str}_{\mathcal{E}/\mathcal{S}} e^{-u\theta^{\mathcal{E}/\mathcal{S}}} \in \Omega^*(M)[u].$$

The proof of the following result is standard and analogous to the proof of Proposition 2.23.

Lemma 6.2. *We have $d_{\pi^*\Omega} \text{Ch}_\mathcal{L}(\mathcal{E}/\mathcal{S}) = 0$ and the corresponding class is $H_{\pi^*\Omega}^*(M)$ is independent of the choice of Clifford connection $\nabla^\mathcal{E}$.*

We also introduce the fiberwise \widehat{A} -genus by $\widehat{A}(TM|B) = \widehat{A}(uR^{M|B}) \in \Omega^*(M)[u]$, where $\widehat{A}(x)$ is the power series defined by

$$\widehat{A}(x) = \det^{1/2} \left(\frac{x/2}{\sinh x/2} \right).$$

Using the above data, we can define on each submersion $\pi^{-1}U_\alpha \rightarrow U_\alpha$ a family of Dirac operators D_α acting on the sections of the bundle \mathcal{E}_α . Locally $D_\alpha = \sum_i c(e^i) (\nabla_\alpha^\mathcal{E})_{e_i}$ where $\{e_i\}$, $\{e^i\}$ are dual bases of $T(M|B)$ and $T^*(M|B)$ respectively. The following is obvious from the definitions.

Lemma 6.3. *The collection $D = \{D_\alpha\}$ defines an element in $\Psi_\mathcal{L}^1(M|B; \mathcal{E})$.*

With respect to the decomposition $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$, the Dirac operator D decomposes as $D = D^+ \oplus D^-$ where

$$D^+ \in \Psi_\mathcal{L}^1(M|B; \mathcal{E}^+, \mathcal{E}^-), \quad D^- \in \Psi_\mathcal{L}^1(M|B; \mathcal{E}^-, \mathcal{E}^+).$$

Classical arguments show that D^+ is fiberwise elliptic and hence the analytical index $\text{ind}(D^+)$ of D^+ is well defined in $K(\Psi_\mathcal{L}^{-\infty}(M|B; \mathcal{E}))$.

Recall that a twisted superconnection \mathbb{A} on the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ is a collection $(\mathbb{A}_\alpha)_{\alpha \in \Lambda}$ of superconnections on the vector bundles $\pi_*\mathcal{E}_\alpha$ over the open sets U_α such that when $U_{\alpha\beta} \neq \emptyset$,

$$(6.1.2) \quad (\pi_*\varphi_{\alpha\beta})^* \mathbb{A}_\beta = \mathbb{A}_\alpha \otimes \text{id} + \text{id} \otimes \nabla_{\alpha\beta}.$$

We say that \mathbb{A} is a Bismut superconnection if each \mathbb{A}_α is. Specifically we have

$$\mathbb{A}_\alpha = u^{-1/2} D_\alpha + \nabla_\alpha^\mathcal{H} - \frac{1}{4} u^{1/2} c(T^\mathcal{H})$$

where $T^{\mathcal{H}}$ is defined in (2.4.1) and we refer to [7] for the precise definition of the Bismut superconnection \mathbb{A}_α .

The facts that $\nabla^{\mathcal{H}}$ is a connection on the descent datum, that $D \in \Psi_{\mathcal{L}}^1(M|B, \mathcal{E})$ and that $c(T^{\mathcal{H}}) \in \Omega^2(B, \Psi^0(M|B, \mathcal{E}))$, imply that the conditions (6.1.2) are satisfied. From now on \mathbb{A} will denote the Bismut superconnection.

We will also consider as before the rescaled Bismut superconnection \mathbb{A}_s defined by

$$(\mathbb{A}_s)_\alpha = \mathbb{A}_{\alpha,s} := s^{1/2} u^{-1/2} D_\alpha + \nabla_\alpha^{\mathcal{H}} - \frac{1}{4} u^{1/2} s^{-1/2} c(T^{\mathcal{H}})$$

where s is either a positive number or a multiple of the formal variable u . Denote by $\theta_\alpha^{\mathbb{A}_s}$ the curvature of the rescaled superconnection $\mathbb{A}_{\alpha,s}$. In particular we have forms $u\theta_{\alpha,u^{-1}}^{\mathbb{A}} \in \Omega^*(U_\alpha, \Psi(\pi^{-1}U_\alpha|U_\alpha, \mathcal{E}_\alpha))[u^{1/2}]$.

Proposition 6.4. *There exists a form $u\theta^{\mathbb{A}} \in \Omega^*(B, \Psi_{\mathcal{L}}(M|B; \mathcal{E}))[u^{1/2}]$ such that*

$$u\theta^{\mathbb{A}}|_{U_\alpha} = u(\theta_{\alpha,u^{-1}}^{\mathbb{A}} + \pi^*\omega_\alpha).$$

Proof. Recall that the curvature of $\nabla_{\alpha\beta}$ is equal to $\omega_\alpha - \omega_\beta$. Therefore from the equation (6.1.2) we obtain

$$\phi_{\alpha\beta}(u\theta_{\beta,u^{-1}}^{\mathbb{A}}) = u(\theta_{\alpha,u^{-1}}^{\mathbb{A}} + \pi^*\omega_\alpha - \pi^*\omega_\beta).$$

The statement of the Proposition follows. \square

Notice that $u\theta^{\mathbb{A}} = D^2 +$ forms of degree > 0 . Therefore we can define

$$e^{u\theta^{\mathbb{A}}} \in \Omega^*(B, \Psi_{\mathcal{L}}^{-\infty}(M|B; \mathcal{E}))[u^{1/2}]$$

by the usual Duhamel formula.

Note that again the parity considerations as in the finite dimensional case show that the coefficients for the nonintegral powers of u are odd with respect to the grading and hence have a vanishing supertrace.

6.2. Projective superconnections on foliations.

Assume now that we are in the situation of Section 5 with the \mathcal{G} -submersion $P \rightarrow \mathcal{G}^{(0)}$. Assume that the twisted Clifford module \mathcal{E} is \mathbb{Z}_2 graded and

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

is the Dirac type family of operators. We construct for every $k \in \mathbb{N}^*$ the Bismut superconnection $\mathbb{A}^{(k)}$ on the fibration $\mathcal{P}^{(k)} \times \Delta^k \rightarrow \mathcal{G}^{(k)} \times \Delta^k$. Proposition 6.4 applied with $M = \mathcal{P}^{(k)} \times \Delta^k$ and $B = \mathcal{G}^{(k)} \times \Delta^k$ gives the form $u\theta^{\mathbb{A}^{(k)}}$.

We similarly produce a collection of morphisms in the following way. For a cochain $c \in (\mathcal{C}_\Delta(\mathcal{G})^\bullet[u], d_{\Omega(L)}^t)$, $c = \{c_k\}$ we have

$$\Phi^{\mathbb{A}}(c) = (-1)^{N(c)} \sum_{j=-\infty}^m (u^*)^j \phi_j^{\mathbb{A}}(c)$$

where $m = \lfloor \frac{1}{2} \deg c \rfloor$, $N(c) = \frac{\deg c(\deg c + 1)}{2}$, $\phi_j^{\mathbb{A}}(c) = \sum (\phi_j^{\mathbb{A}})^{kl}(c_k)$ and $(\phi_j^{\mathbb{A}})^{kl}$ is the element of $C^k(\mathcal{G}, C_{entire}^l(\Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)})))$ defined for $k + l = \deg c - 2j$ by

$$(6.2.1) \quad (\phi_j^{\mathbb{A}})^{kl}(c_k)(A_0, \dots, A_l) := \int_{\Delta^l} \left\langle (u^*)^{-j} c_k, \text{Str} \left(A_0 e^{-\sigma_0 u \theta^{\mathbb{A}}(k)} [\mathbb{A}^{(k)}, A_1] e^{-\sigma_1 u \theta^{\mathbb{A}}(k)} \dots e^{-\sigma_{l-1} u \theta^{\mathbb{A}}(k)} [\mathbb{A}^{(k)}, A_l] e^{-\sigma_l u \theta^{\mathbb{A}}(k)} \right) \right\rangle d\sigma_1 \dots d\sigma_l$$

Note that only integral powers of u appear in this formula since the coefficients of half-integral powers of u involve the supertrace of odd operators. Notice that this formula extends the usual untwisted one and yields an entire cyclic cocycle, see for instance [3].

Consider now the sheaf of algebras $\mathcal{A} \subset \Psi_{\mathcal{L}}^0(\pi, \mathcal{E})$ on $\mathcal{G}^{(0)}$ given by:

$$U \text{ open subset of } \mathcal{G}^{(0)} \longrightarrow \mathcal{A}(U) = \{A \in \Psi_{\mathcal{L}}^0(\pi, \mathcal{E})(U) \mid [D, A] \in \Psi_{\mathcal{L}}^0(\pi, \mathcal{E})\}.$$

When no confusion arises we will also denote by \mathcal{A} the algebra of global sections, which is a subalgebra of $\Psi_{\mathcal{L}}^0(P|\mathcal{G}^{(0)}, \mathcal{E})$. This is a \mathcal{G} -subsheaf of $\Psi_{\mathcal{L}}^0(\pi, \mathcal{E})$. Consider now the complex $C^\bullet(\mathcal{G}, \text{CC}_{\text{entire}}^\bullet(\mathcal{A}))$. Recall that this is a total complex of the bicomplex given in the bidegree l, k by $\text{CC}^l(\mathcal{A}^{(k)})$, where $\mathcal{A}^{(k)} = (\text{pr}_0^k)^* \mathcal{A}(\mathcal{G}^{(k)}) \subset \Psi_{\mathcal{L}}^0(\mathcal{P}^{(k)}|\mathcal{G}^{(k)}, \mathcal{E}^{(k)})$. The cochains $(\Phi_{\mathbb{A}}^{(k)})^l(c)$, defined by the same formulas, extend to define elements in $C^\bullet(\mathcal{G}, \text{CC}_{\text{entire}}^\bullet(\mathcal{A}))$.

We therefore obtain a morphism:

$$\Phi_{\mathbb{A}}: (\mathcal{C}_{\Delta}(\mathcal{G})^\bullet, d_{\Omega(L)}^t) \longrightarrow C^\bullet(\mathcal{G}, \text{CC}_{\text{entire}}^\bullet(\mathcal{A})).$$

For every $0 < s \leq 1$ we also have the rescaled superconnection \mathbb{A}_s and the corresponding morphism $\Phi_{\mathbb{A}_s}$.

We will now apply this construction of the morphism $\Phi_{\mathbb{A}}$ to the groupoid $\mathcal{G} \times \mathbb{R}^+$. Let s be the coordinate in \mathbb{R}^+ and d_s the de Rham differential on \mathbb{R}^+ . Equip every submersion $\mathcal{P}^{(k)} \times \Delta^k \rightarrow \mathcal{G}^{(k)} \times \Delta^k$ with the superconnection $d_s + \mathbb{A}_s$. We obtain the morphism

$$\Phi_{d_s + \mathbb{A}_s}: (\mathcal{C}_{\Delta}^{vc}(\mathcal{G} \times \mathbb{R}^+)^\bullet, u^* d_s^t + d_{\Omega(L)}^t) \rightarrow C^\bullet(\mathcal{G}, \text{CC}_{\text{entire}}^\bullet(\mathcal{A})).$$

Here $\mathcal{C}_{\Delta}^{vc}(\mathcal{G} \times \mathbb{R}^+)^\bullet$ denotes the currents with fiberwise compact support on (trivial) fibrations $\mathcal{G}^{(k)} \times \Delta^k \times \mathbb{R}^+ \rightarrow \mathcal{G}^{(k)} \times \Delta^k$, and \mathcal{A} is viewed as sections on $\mathcal{G} \times \mathbb{R}^+$ constant in \mathbb{R}^+ -direction.

For every $\tau \in \mathbb{R}^+$ let $i_\tau: \Omega^\bullet(\mathcal{G} \times \mathbb{R}^+) \rightarrow \Omega^{\bullet-1}(\mathcal{G})$ be the map defined by

$$i_\tau(\alpha + \beta ds) = \beta|_{s=\tau}.$$

By transposition it induces a map

$$i_\tau^t: \mathcal{C}_{\Delta}(\mathcal{G})^\bullet \longrightarrow \mathcal{C}_{\Delta}^{vc}(\mathcal{G} \times \mathbb{R}^+)^{\bullet+1}$$

Composing it with $\Phi_{d_s + \mathbb{A}_s}$ we obtain the transgression map

$$\Psi_{\mathbb{A}_s} := \Phi_{d_s + \mathbb{A}_s} \circ i_\tau^t: \mathcal{C}_{\Delta}(\mathcal{G})^\bullet \longrightarrow C^{\bullet+1}(\mathcal{G}, \text{CC}_{\text{entire}}^\bullet(\mathcal{A})).$$

The following lemma is the exact twisted version of a classical relation, it can be verified by a direct computation:

Lemma 6.5.

$$u^* \frac{d}{ds} \Phi_{\mathbb{A}_s} = (b \pm \partial + u^* B) \circ \Psi_{\mathbb{A}_s} + \Psi_{\mathbb{A}_s} \circ d_{\Omega(L)}^t$$

We now fix a cocycle $c \in (\mathcal{C}_\Delta(\mathcal{G})^\bullet[u], d_{\Omega(L)}^t)$, and write $\phi_j(s)$ for $\phi_j^{\mathbb{A}_s}(c)$ and $\phi_j^{kl}(s)$ for $(\phi_j^{\mathbb{A}_s})^{kl}(c)$. So $\phi_j^{pq}(s) \in C^p(\mathcal{G}, C^q(\mathcal{A}))$, $p + q = \deg c - 2j$. Write similarly

$$\Psi_{\mathbb{A}_s}(c) = (-1)^{N(c)} \sum_{j=-\infty}^{m+1} (u^*)^j \psi_j(s)$$

Then

$$\frac{d}{ds} \phi_j(s) = (b \pm \partial) \psi_{j+1}(s) + B \psi_j(s)$$

We also use the notations of Appendix C, and we can state the following consequence of Lemma C.1 and Lemma C.2:

Proposition 6.6. *For a current c there exist constants M and $C = C(j)$ such that*

$$|\phi_j^{kl}(s)(A_0, A_1, \dots, A_l)| \leq C \mu(A_0) \nu_n(A_1) \dots \nu_n(A_l) s^{-M-j/2}$$

and

$$|\psi_j^{kl}(s)(A_0, A_1, \dots, A_l)| \leq C \mu(A_0) \nu_n(A_1) \dots \nu_n(A_l) s^{-M-(j+1)/2}$$

where $\nu_n(A) = \max\{\|(1+D^2)^i A(1+D^2)^{-i}\| \mid |i| \leq n\} + \max\{\|(1+D^2)^i [D, A](1+D^2)^{-i}\| \mid |i| \leq n\}$.

Proof. First consider the expression $e^{-\sigma u \theta_s^{\mathbb{A}_s}(k)}$. By the Duhamel expansion formula, this is a finite sum (polynomial in u) of the form $\sum u^i P_i$. Each term of this expansion can be bounded by using Lemma C.1. We obtain that for sufficiently large r and for each i the (fiberwise) norm of $(1+D^2)^{-r} P_i$ is bounded uniformly in s, σ . If we assume also that $\sigma \geq \frac{1}{l}$ then we have:

$$(6.2.2) \quad \|(1+D^2)^r P_i\| \leq C s^{-r-M},$$

for some universal constant M . Consider now $\phi_j^{kl}(s)(A_0, A_1, \dots, A_l)$. As in Lemma C.1 it is sufficient to bound the integrand assuming $\sigma_i \geq 1/l$ and then add the estimates. The expansion of the supertrace in powers of u contains u^j multiplied by s to the power $-j/2$ or higher and by the integral which can be bounded as in Lemmas C.1, C.2 using the estimate (6.2.2). The result follows immediately. \square

The entire cocycle $\Phi_{\mathbb{A}_s}(c)$ can indeed be embodied by a periodic one. Set for $p \gg 1$ sufficiently large

$$(6.2.3) \quad T_p \Phi_{\mathbb{A}_s}(c) := (-1)^{N(c)} \left(\sum_{j=-p}^m (u^*)^j \phi_j(s) - (u^*)^{-p} B \int_0^s \psi_{-p-1}(t) dt \right).$$

We use the notation $T_p \Phi_{\mathbb{A}}$ for $T_p \Phi_{\mathbb{A}_1}$.

Proposition 6.7. *For every $0 < s \leq 1$ and sufficiently large p , $T_p \Phi_{\mathbb{A}_s}(c)$ is a cocycle in the complex $C^\bullet(\mathcal{G}, \text{CC}_{\text{per}}^\bullet(\mathcal{A}))$. Its cohomology class does not depend on choices of s and p .*

Proof. We note that Proposition 6.6 implies that for sufficiently large p ($p > 2M$ in the notations of the proposition) $\phi_{-p}(s) \rightarrow 0$ as $s \rightarrow 0$, and the integral $\int_0^s \psi_{-p-1}(t)dt$ converges, when evaluated on collections of operators A_i with uniformly bounded norms μ, ν_n . It follows that

$$\phi_{-p}(s) = (b \pm \partial) \int_0^s \psi_{-p}(t)dt + B \int_0^s \psi_{-p-1}(t)dt$$

We also have identities

$$(b \pm \partial)\phi_j(s) + B\phi_{j-1}(s) = 0.$$

Together these identities easily imply that $T_p\Phi_{\mathbb{A}_s}(c)$ is a cocycle in the complex $C^\bullet(\mathcal{G}, \text{CC}_{per}^\bullet(\mathcal{A}))$ for any s and sufficiently large p . We also have

$$u^* \frac{d}{ds} T_p \Phi_{\mathbb{A}_s}(c) = (-1)^{N(c)} (b \pm \partial + u^* B) \sum_{j=-p}^{m+1} (u^*)^j \psi_j(s)$$

Finally for $q > p$, we have

$$u^* (T_p \Phi_{\mathbb{A}_s}(c) - T_q \Phi_{\mathbb{A}_s}(c)) = (-1)^{N(c)} (b \pm \partial + u^* B) \sum_{j=-q}^{-p-1} (u^*)^j \psi_j(s)$$

□

Now, applying the map

$$H^\bullet(C^\bullet(\mathcal{G}, \text{CC}^\bullet(\mathcal{A})) \longrightarrow \text{HC}^\bullet(\mathcal{A} \rtimes \mathcal{G}),$$

constructed in [28], see also [15], together with the map

$$\text{HC}^\bullet(\mathcal{A} \rtimes \mathcal{G}) \longrightarrow \text{HC}^\bullet(\mathcal{A}^{\mathcal{G}}),$$

to the class of the above cocycle $[T_p\Phi_{\mathbb{A}}(c)]$, we eventually end up with the class $\Lambda_{\mathbb{A}}(c)$ which thus lives in the periodic cyclic cohomology $\text{HC}_{per}^\bullet(\mathcal{A}^{\mathcal{G}})$.

Proposition 6.8. *Let $\iota: \Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}, \mathcal{E}) \rightarrow \mathcal{A}$ be the inclusion. Then*

$$[\iota^*(T_p\Phi_{\mathbb{A}}(c))] = [\Phi^{\mathcal{E}, \gamma}(c)] \text{ and } \iota^*\Lambda_{\mathbb{A}}(c) = \Lambda^{\mathcal{E}, \gamma}(c).$$

See Remark 6.3 for the definition of $\Lambda^{\mathcal{E}, \gamma}(c)$.

Proof. First notice that the second equality is an immediate consequence of the first. To prove the first notice that applying the construction of the morphism $T_p\Phi_{\mathbb{A}}$ to the groupoid $\mathcal{G} \times \mathbb{R}^+$, as in discussion preceding Lemma 6.5, one shows that the class of $T_p\Phi_{\mathbb{A}}(c)$ is independent of superconnection \mathbb{A} . Moreover, when restricted to the algebra of smoothing operators one can homotop \mathbb{A} to the connection $\nabla^{\mathcal{H}}$ and obtain that the cocycles $\iota^*T_p\Phi_{\mathbb{A}}(c)$ and $\Phi^{\mathcal{E}, \gamma}(c)$ are cohomologous. □

We are now in position to compute our higher index in terms of the superconnection \mathbb{A} .

Theorem 6.9. *For any cocycle $c \in (\mathcal{C}_{\Delta}(\mathcal{G})^\bullet[u^*], d_{\Omega(L)}^t)$, the higher twisted index is given by:*

$$\text{ind}_c(D^+) = \left\langle (u^*)^{-\frac{\deg c}{2}} \Lambda_{\mathbb{A}}(c), [1] \right\rangle.$$

Proof. Consider the twisted bundle $\tilde{\mathcal{E}} = \mathcal{E} \oplus \mathcal{E}$ with the grading given by $\Gamma = \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}$. The algebra $\Psi_{\mathcal{L}}(P|\mathcal{G}^{(0)}; \tilde{\mathcal{E}})$ of operators on $\tilde{\mathcal{E}}$ is naturally \mathbb{Z}_2 graded. For an operator or (super) connection K on \mathcal{E} set $\tilde{K} = K \oplus K$; so for example $\tilde{\mathbb{A}} = \mathbb{A} \oplus \mathbb{A}$, $\tilde{\nabla}^{\mathcal{H}} = \nabla^{\mathcal{H}} \oplus \nabla^{\mathcal{H}}$, etc. In particular $\tilde{\mathcal{A}}$ is a sheaf defined by

$$\tilde{\mathcal{A}}(U) = \{A \in \Psi_{\mathcal{L}}^0(\pi, \tilde{\mathcal{E}})(U) \mid [\tilde{D}, A] \in \Psi_{\mathcal{L}}^0(\pi, \tilde{\mathcal{E}})\}.$$

Let $F = D(1 + D^2)^{-1/2}$. Then it is immediate that $F \in \Psi_{\mathcal{L}}^0(P|\mathcal{G}^{(0)}; \tilde{\mathcal{E}})^{\mathcal{G}}$ is odd with respect to the grading Γ and fiberwise elliptic. Construct the invertible operator $U_D \in M_2\left(\Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}; \tilde{\mathcal{E}})^{\mathcal{G}}\right)$ by the same formula as before. Namely, choose a parametrix Q for F . Let $S_0 = 1 - QF$, $S_1 = 1 - FQ$. Then set $U_D = \begin{bmatrix} S_0 & -(1 + S_0)Q \\ F & S_1 \end{bmatrix}$. With such a choice the inverse is given by an explicit formula $U_D^{-1} = \begin{bmatrix} S_0 & (1 + S_0)Q \\ -F & S_1 \end{bmatrix}$. Set $P_D = U_D^{-1} \begin{bmatrix} 1_{\mathcal{E}} & 0 \\ 0 & 0 \end{bmatrix} U_D$.

The choices in the constructions can be made so that we have $(P_D)^{\pm} = P_{D^{\pm}}$, see Definition 4.8. We then compute

$$\begin{aligned} \text{ind}_c(D^+) &:= \langle (u^*)^{-\frac{\deg c}{2}} \Lambda^{\mathcal{E}}(c), \text{ch}(\text{ind } D^+) \rangle = \\ &= \frac{1}{2} \left(\langle (u^*)^{-\frac{\deg c}{2}} \Lambda^{\mathcal{E}}(c), \text{Ch}(\text{ind } D^+) \rangle - \langle (u^*)^{-\frac{\deg c}{2}} \Lambda^{\mathcal{E}}(c), \text{Ch}(\text{ind } D^-) \rangle \right) = \\ &= \frac{1}{2} \left\langle (u^*)^{-\frac{\deg c}{2}} \Lambda^{\tilde{\mathcal{E}}, \Gamma}(c), \left(\text{Ch } P_D - \text{Ch} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathcal{E}} \end{bmatrix} \right) \right) \right\rangle = \\ &= \frac{1}{2} \left\langle (u^*)^{-\frac{\deg c}{2}} \Lambda_{\tilde{\mathbb{A}}}(c), \left(\text{Ch } P_D - \text{Ch} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathcal{E}} \end{bmatrix} \right) \right) \right\rangle, \end{aligned}$$

where the last equality is a consequence of Proposition 6.8. Since U_D is an even element of the algebra $\tilde{\mathcal{A}}$, $\text{Ch}(P_D) = \text{Ch} \left(\begin{bmatrix} 1_{\mathcal{E}} & 0 \\ 0 & 0 \end{bmatrix} \right)$ in $\text{HC}_0^-(\mathcal{A})$, and therefore

$$\langle (u^*)^{-\frac{\deg c}{2}} \Lambda^{\mathcal{E}}(c), \text{Ch}(\text{ind } D^+) \rangle = \frac{1}{2} \left\langle (u^*)^{-\frac{\deg c}{2}} \Lambda_{\tilde{\mathbb{A}}}(c), \left(\text{Ch} \left(\begin{bmatrix} 1_{\mathcal{E}} & 0 \\ 0 & 0 \end{bmatrix} \right) - \text{Ch} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathcal{E}} \end{bmatrix} \right) \right) \right\rangle.$$

To complete the proof, we observe that

$$\begin{aligned} \left\langle (u^*)^{-\frac{\deg c}{2}} \Lambda_{\tilde{\mathbb{A}}}(c), \text{Ch} \left(\begin{bmatrix} 1_{\mathcal{E}} & 0 \\ 0 & 0 \end{bmatrix} \right) \right\rangle &= \\ &= - \left\langle (u^*)^{-\frac{\deg c}{2}} \Lambda_{\tilde{\mathbb{A}}}(c), \text{Ch} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathcal{E}} \end{bmatrix} \right) \right\rangle = \left\langle (u^*)^{-\frac{\deg c}{2}} \Lambda_{\mathbb{A}}(c), [1] \right\rangle \end{aligned}$$

□

Theorem 6.9 allows by the standard method, see [36], to deduce the higher twisted index theorem for foliations. For the sake of clarity we gather in the next paragraph the main steps of the proof with the explicit formula we get.

6.3. Proof of the higher index theorem. The HKR-Connes morphism $\chi: (\mathcal{C}^\bullet(Z), d) \rightarrow (\text{CC}^\bullet(C_c^\infty(Z), B)$ for an arbitrary manifold Z is given by formula

$$\chi(\xi)(a_0, a_1, \dots, a_k) = \frac{1}{k!} \langle \xi, a_0 da_1 \dots da_k \rangle$$

for a current ξ and then extended u^* -linearly. It induces for our proper and free \mathcal{G} -compact manifold P , a morphism, denoted

$$\chi_*: C^\bullet(\mathcal{G}, \mathcal{C}^\bullet(P)) \longrightarrow C^\bullet(\mathcal{G}, \text{CC}^\bullet(C_c^\infty(P))).$$

Composing the morphism χ with the integration quasi-isomorphism

$$\int_\Delta: \mathcal{C}_\Delta(\mathcal{P}) \longrightarrow C^\bullet(\mathcal{G}, \mathcal{C}^\bullet(P)),$$

we obtain a morphism

$$\Xi := \chi_* \circ \int_\Delta: \mathcal{C}_\Delta(\mathcal{P}) \longrightarrow C^\bullet(\mathcal{G}, \text{CC}^\bullet(C_c^\infty(P)))$$

Since the action of \mathcal{G} on P is proper the inclusion

$$\mathcal{C}^\bullet(P/\mathcal{G}) = \mathcal{C}^\bullet(P)^\mathcal{G} \rightarrow C^\bullet(\mathcal{G}, \mathcal{C}^\bullet(P))$$

is a quasiisomorphism. Let $R: C^\bullet(\mathcal{G}, \mathcal{C}^\bullet(P)) \rightarrow \mathcal{C}^\bullet(P/\mathcal{G})$ be a (right) inverse quasiisomorphism constructed explicitly using for instance Lemma 1.1. Composing the morphism induced by χ_* on cohomology with the composite map

$$H^\bullet(C^\bullet(\mathcal{G}, \text{CC}^\bullet(C_c^\infty(P)))) \xrightarrow{\alpha_P} \text{HC}^\bullet(C_c^\infty(P) \rtimes \mathcal{G}) \xrightarrow{\beta_P} \text{HC}^\bullet(C_c^\infty(P/\mathcal{G}))$$

we obtain a map $\widehat{\chi}: H^\bullet(C^\bullet(\mathcal{G}, \mathcal{C}^\bullet(P))) \longrightarrow \text{HC}^\bullet(C_c^\infty(P/\mathcal{G}))$ which is more precisely induced on cohomology by $\beta_P \circ \alpha_P \circ \chi_*$. We also have the similar maps:

$$H^\bullet(C^\bullet(\mathcal{G}, \text{CC}^\bullet(\mathcal{A}))) \xrightarrow{\alpha_\mathcal{A}} \text{HC}^\bullet(\mathcal{A} \rtimes \mathcal{G}) \xrightarrow{\beta_\mathcal{A}} \text{HC}^\bullet(\mathcal{A}^\mathcal{G}),$$

as well as the corresponding ones for the ideal $\Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}; \mathcal{E})$ that we denote by $\alpha_{\Psi^{-\infty}}$ and $\beta_{\Psi^{-\infty}}$. Recall that $i: \Psi_{\mathcal{L}}^{-\infty}(P|\mathcal{G}^{(0)}; \mathcal{E}) \hookrightarrow \mathcal{A}$ and set $I: C_c^\infty(P) \hookrightarrow \mathcal{A}$, then we have

$$i^* \circ (\beta_\mathcal{A} \circ \alpha_\mathcal{A}) = (\beta_{\Psi^{-\infty}} \circ \alpha_{\Psi^{-\infty}}) \circ i^* \text{ and } I^* \circ (\beta_\mathcal{A} \circ \alpha_\mathcal{A}) = (\beta_P \circ \alpha_P) \circ I^*.$$

Lemma 6.10. *Let ϱ be a cocycle in $C^\bullet(\mathcal{G}, \mathcal{C}^\bullet(P))$. Then*

$$\langle \widehat{\chi}([\varrho]), [1] \rangle = \int_{P/\mathcal{G}} R(\varrho) := \langle R(\varrho), 1 \rangle.$$

Proof. Since both sides of the equality depend only on the cohomology class $[\varrho]$, we can assume that ϱ is \mathcal{G} -invariant, in which case the result is clear. \square

Let now $\widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \in \Omega_\bullet^\Delta(\mathcal{P})$ denote the form whose component in $\Omega^*(\mathcal{P}^{(k)} \times \Delta^k)$ is $\widehat{A}(\mathcal{P}^{(k)}|\mathcal{G}^{(k)}) \text{Ch}_{\mathcal{L}^{(k)}}(\mathcal{E}_\Delta^{(k)}/S)$. This is a closed form in $(\Omega_\bullet^\Delta(\mathcal{P}), d_{\Omega(L)}^t)$. Therefore if c is a cocycle in $(\mathcal{C}_\Delta^\bullet(\mathcal{P})[(u^*)^{-1}, u^*], d_{\Omega(L)}^t)$ then the pairing already described yields a cocycle $\pi^*c \cap (\widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S))$ in the untwisted complex $(\mathcal{C}_\Delta^\bullet(\mathcal{P})[(u^*)^{-1}, u^*], d)$. Hence, we can

integrate it and eventually define, when c is an even cocycle, the higher twisted characteristic number (with the appropriate normalization constant):

$$\left\langle \pi^* c, \widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right\rangle := (2\pi i)^{-l/2} \left(\int_{P/\mathcal{G}} \circ R \circ \int_{\Delta} \right) \left((u^*)^{-\frac{\deg c + l}{2}} (\pi^* c) \cap \widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right).$$

The classical short time limit of the heat operator associated with our Bismut superconnection yields:

Proposition 6.11. *Let c be a cocycle in $(\mathcal{C}_{\Delta}^{\bullet}(\mathcal{G})[(u^*)^{-1}, u^*], d_{\Omega(L)}^t)$. Then*

$$[I^* T_p \Phi_{\mathbb{A}}(c)] = (2\pi i u^*)^{-l/2} \left[\Xi \left((\pi^* c) \cap \widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right) \right].$$

where $l = \dim P - \dim \mathcal{G}^{(0)}$ is the fiberwise dimension and the operator T_p is defined in (6.2.3).

We are now in position to state our main theorem:

Theorem 6.12 (The higher twisted index theorem for foliations). *Let c be an even-degree cocycle in $(\mathcal{C}_{\Delta}^{\bullet}(\mathcal{G})[(u^*)^{-1}, u^*], d_{\Omega(L)}^t)$. Then the higher twisted index is given by:*

$$\text{ind}_c(D^+) = \left\langle \pi^* c, \widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right\rangle.$$

Proof. We simply combine the results of Theorem 6.9, Lemma 6.10 and Proposition 6.11. We can rewrite the expression $\left\langle \pi^* c, \widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right\rangle$ as

$$\begin{aligned} (2\pi i)^{-l/2} \left\langle (R \circ \chi_*^{-1} \circ \Xi) \left((u^*)^{-\frac{\deg c + l}{2}} (\pi^* c) \cap \widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right), 1 \right\rangle \\ = \left\langle (u^*)^{-\frac{\deg c}{2}} (R \circ \chi_*^{-1}) (I^* T_p \Phi_{\mathbb{A}}(c)), 1 \right\rangle. \end{aligned}$$

Applying Proposition 6.11 to $\varrho := \chi_*^{-1}(I^* T_p \Phi_{\mathbb{A}}(c))$ we deduce

$$\begin{aligned} \left\langle \pi^* c, \widehat{A}(P|\mathcal{G}^{(0)}) \text{Ch}_L(\mathcal{E}/S) \right\rangle &= \left\langle (u^*)^{-\frac{\deg c}{2}} (\beta_P \circ \alpha_P) (I^* T_p \phi_{\mathbb{A}}(c)), [1] \right\rangle \\ &= \left\langle (u^*)^{-\frac{\deg c}{2}} (\beta_{\mathcal{A}} \circ \alpha_{\mathcal{A}}) (T_p \phi_{\mathbb{A}}(c)), [1] \right\rangle = \left\langle (u^*)^{-\frac{\deg c}{2}} \Lambda_{\mathbb{A}}(c), [1] \right\rangle. \end{aligned}$$

Therefore, the proof is complete since it is reduced to Theorem 6.9. \square

APPENDIX A. TWISTED COHOMOLOGY

For a smooth manifold M let $\Omega \in \Omega^3(M)$ be a closed 3-form. Denote by u a formal variable of degree -2 . The twisted de Rham complex is defined as the complex $\Omega^*(M)[u]$ with the differential $d_{\Omega} = u d + u^2 \Omega \wedge \cdot$. Its homology is denoted by $H_{\Omega}^{\bullet}(M)$. Note that if $\Omega' = \Omega + d\eta$ is cohomologous to Ω then the complexes $(\Omega^*(M)[u], d_{\Omega})$ and $(\Omega^*(M)[u], d_{\Omega'})$ are isomorphic via the isomorphism

$$(A.0.1) \quad I_{\eta}: \xi \mapsto e^{-u\eta} \wedge \xi$$

We have more precisely [4]:

Lemma A.1. *The map induced by I_η on cohomology depends only on the class of η in $\Omega^2(M)/d\Omega^1(M)$.*

The identity $d_{\Omega+\Omega'}(\xi \wedge \eta) = d_\Omega \xi \wedge \eta + (-1)^{|\xi|} \xi \wedge d_{\Omega'} \eta$ implies that the product of forms induces the product $H_\Omega^\bullet(M) \otimes H_{\Omega'}^\bullet(M) \rightarrow H_{\Omega+\Omega'}^\bullet(M)$ and in particular endows $H_\Omega^\bullet(M)$ with the structure of $H^\bullet(M)[u]$ -module.

We will also need to consider the following situation. Let $\pi: M \rightarrow B$ be an oriented fibration with compact fibers. Then we have an integration along the fibers map $\int_{M|B}: \Omega^*(M) \rightarrow \Omega^{*-k}(B)$ such that $\int_{M|B} d\eta = d \int_{M|B} \eta$, where $k = \dim M - \dim B$. Let $\Omega \in \Omega^3(B)$ be a closed form. Then $\int_{M|B} \pi^* \Omega \wedge \eta = \Omega \wedge \int_{M|B} \eta$ and therefore

$$\int_{M|B} d_{\pi^* \Omega} \eta = d_\Omega \int_{M|B} \eta.$$

Hence we obtain a chain map $\int_{M|B}: (\Omega^*(M)[u], d_{\pi^* \Omega})_\bullet \rightarrow (\Omega^*(B)[u], d_\Omega)_{\bullet-k}$.

Let \mathcal{L} be a gerbe. The choice of connection defines a closed 3-form Ω . We can therefore consider the complex $(\Omega^*(M)[u], d_\Omega)$. A different choice of connection leads to a different complex $(\Omega^*(M)[u], d_{\Omega'})$. Lemma 2.11 together with Lemma A.1 implies that there exists however a canonical isomorphism I of the homologies of these complexes. We thus denote this (canonical isomorphism class of the) homology space by $H_\mathcal{L}^\bullet(M)$.

A.0.1. In the sequel it will be convenient to also work with regular currents and twisted de Rham homology. For a manifold M , $\mathcal{C}_l(M)$, the space of smooth currents of degree l , is given by $\mathcal{C}_l(M) = \Omega^{\dim M - l}(M, or)$, where or is the orientation bundle. It is a subspace of the vector space of distributions on $\Omega^l(M)$ with the pairing defined by the formula

$$\langle \xi, \omega \rangle = \int \xi \wedge \omega.$$

The differential $d^t: \mathcal{C}_l(M) \rightarrow \mathcal{C}_{l-1}(M)$ is the transposed of the de Rham differential up to a sign:

$$\langle d^t \xi, \omega \rangle = (-1)^{\deg \xi + 1} \langle \xi, d\omega \rangle, \quad \xi \in \mathcal{C}_*(M), \omega \in \Omega^*(M).$$

We also have an operation of cap product $\mathcal{C}_l(M) \otimes \Omega^m(M) \rightarrow \mathcal{C}_{l-m}(M)$ defined by

$$\langle \xi \cap \alpha, \omega \rangle = \langle \xi, \alpha \wedge \omega \rangle, \quad \xi \in \mathcal{C}_*(M), \alpha, \omega \in \Omega^*(M).$$

With these conventions we have:

$$\xi \cap (\alpha \wedge \beta) = (\xi \cap \alpha) \cap \beta, \quad \langle \xi \cap \alpha, 1 \rangle = \langle \xi, \alpha \rangle$$

and

$$(A.0.2) \quad d^t(\xi \cap \alpha) = (d^t \xi) \cap \alpha + (-1)^{\deg \xi} \xi \cap d\alpha$$

Introduce the formal variable u^* of degree 2. Form now the complexes $\mathcal{C}^\bullet(M) = \mathcal{C}_*(M)[u^*]$ with the differential of degree 1 given by $u^* d^t$. If Ω is a form of degree 3 we can also introduce the differential d_Ω^t by the formula

$$(A.0.3) \quad d_\Omega^t \xi = u^* d^t \xi - (u^*)^2 \Omega \cap \xi = u^* d^t \xi + (-1)^{\deg \xi + 1} (u^*)^2 \xi \cap \Omega.$$

We define a complex-valued pairing between $\mathcal{C}^\bullet(M)[(u^*)^{-1}]$ and $\Omega^*(M)[u]$ by

$$\langle (u^*)^k \xi, u^l \omega \rangle = \begin{cases} \langle \xi, \omega \rangle & \text{if } k + l = 0 \\ 0 & \text{otherwise} \end{cases}$$

The cap-product extends by

$$((u^*)^k \xi) \cap (u^l \omega) = (u^*)^{k+l} (\xi \cap \omega).$$

We then have

$$(u^* d^t)(\xi \cap \alpha) = (d_\Omega^t \xi) \cap \alpha + (-1)^{\deg \xi} \xi \cap d_\Omega \alpha.$$

APPENDIX B. REVIEW OF THE MAP Φ

If \mathcal{A} is a sheaf of algebras over \mathcal{G} , one can form a \mathcal{G} -sheaf $CC^\bullet(\mathcal{A})$, defined as the sheafification of the presheaf $U \mapsto CC^*(\Gamma_c(U; \mathcal{A}))$. From now on assume that \mathcal{A} is a c-soft sheaf and we refer to such an \mathcal{A} as a \mathcal{G} -algebra. We can define a complex $C^\bullet(\mathcal{G}, CC^\bullet(\mathcal{A}))$. On the other hand one can form a cross-product algebra $\mathcal{A} \rtimes \mathcal{G}$.

Assume $\lambda \in \Gamma(\mathcal{G}^{(0)}; CC^\bullet(\mathcal{A}))$ is \mathcal{G} -invariant. Then one can construct a cyclic cocycle c_λ on $\mathcal{A} \rtimes \mathcal{G}$ as follows. Let $Z^{(k)} = \{(\gamma_1, \dots, \gamma_k) \in \mathcal{G}^{(k)} \mid r(\gamma_1) = s(\gamma_k) \in \mathcal{G}^{(0)}\}$ be the subset of $\mathcal{G}^{(k)}$. Let $\pi: Z^{(k)} \rightarrow \mathcal{G}^{(0)}$ be the map given by

$$\pi(\gamma_1, \dots, \gamma_k) = r(\gamma_1).$$

Note that we have the following sequence of maps of vector spaces:

$$\Gamma_c(\mathcal{G}^k; (r^* \mathcal{A})^{\boxtimes k}) \rightarrow \Gamma_c(Z^{(k)}; (r^* \mathcal{A})^{\boxtimes k}) \rightarrow \Gamma_c(Z^{(k)}; (\pi^* \mathcal{A})^{\otimes k}) \rightarrow \Gamma_c(\mathcal{G}^{(0)}; \mathcal{A}^{\otimes k})$$

Here the first map is induced by the inclusion $Z^{(k)} \hookrightarrow \mathcal{G}^k$. The second map is induced by the isomorphism of sheaves $(r^* \mathcal{A})^{\boxtimes k} \cong (\pi^* \mathcal{A})^{\otimes k}$ on $Z^{(k)}$ given by \mathcal{G} action on \mathcal{A} . Finally the third map is induced by the map π . Denote the composition of these maps by M_k . Then we can define the cyclic cocycle c_λ by

$$c_\lambda(f_0, f_1, \dots, f_k) = \lambda(M_{k+1}(f_0 \otimes f_1 \otimes \dots \otimes f_k)).$$

One can extend this construction to produce a morphism of complexes

$$(B.0.1) \quad \Phi: C^\bullet(\mathcal{G}, CC^\bullet(\mathcal{A})) \rightarrow CC^\bullet(\mathcal{A} \rtimes \mathcal{G})$$

This map has been constructed in [28], based on the constructions in [15], compare also [53], [32], [33]. An important particular case is when \mathcal{A} is the sheaf $C^\infty(B)$ of smooth functions on B . In this case there is a canonical morphism of complexes of sheaves $\iota: \Omega_*(B) \rightarrow CC^*(C_0^\infty(B))$ defined on the current c of degree m by

$$(B.0.2) \quad \iota(c)(a_0, a_1, \dots, a_m) = \frac{1}{m!} \langle c, a_0 da_1 \dots da_m \rangle.$$

Composing ι with the map in (2.3.1) one obtains (up to chain homotopy) Connes' map [22, 25], also denoted by Φ :

$$(B.0.3) \quad \Phi: C^*(\mathcal{G}, \Omega_*(B)) \rightarrow CC^*(C_0^\infty(\mathcal{G})).$$

We will use the following properties of the map Φ , compare [28]:

- If \mathcal{A}, \mathcal{B} two \mathcal{G} -algebras and $f: \mathcal{A} \rightarrow \mathcal{B}$ is a \mathcal{G} -homomorphism, we get a natural induced homomorphism, which we also denote by f , from $\mathcal{A} \rtimes \mathcal{G}$ to $\mathcal{B} \rtimes \mathcal{G}$. Also the induced map $f^*: CC^*(\mathcal{B}) \rightarrow CC^*(\mathcal{A})$ is \mathcal{G} -equivariant, and hence defines a map, also denoted by f^* , from $C^*(\mathcal{G}, CC^*(\mathcal{B}))$ to $C^*(\mathcal{G}, CC^*(\mathcal{A}))$. Then the following diagram commutes:

$$\begin{array}{ccc} C^\bullet(\mathcal{G}, CC^\bullet(\mathcal{B})) & \xrightarrow{\Phi} & CC^\bullet(\mathcal{B} \rtimes \mathcal{G}) \\ f^* \downarrow & & \downarrow f^* \\ C^\bullet(\mathcal{G}, CC^\bullet(\mathcal{A})) & \xrightarrow{\Phi} & CC^\bullet(\mathcal{A} \rtimes \mathcal{G}) \end{array}$$

- If $\lambda \in \Gamma(\mathcal{G}^{(0)}; CC^\bullet(\mathcal{A}))$ is \mathcal{G} -invariant then

$$\Phi(\lambda) = c_\lambda$$

APPENDIX C. SOME JLO ESTIMATES FOR COMPLETE MANIFOLDS

Let $\|\cdot\|$ denote the operator norm of $BL^2(N, E)$.

Lemma C.1. *Let N be a complete Riemannian manifold and D a Dirac type operator on N acting on a Clifford module E . Assume that the V_i ($0 \leq i \leq l$) are pseudodifferential operators on P such that*

$$\|V_i\|_{v_i, t} = \|(1 + D^2)^{-t/2} V_i (1 + D^2)^{(v_i + t)/2}\| < \infty, \quad v_i = \text{ord } V_i$$

for every $t \in \mathbb{R}$ and every i . Let $q = \max\{\sum_{i=0}^l \text{ord } V_i, 0\}$. Then for $0 < s \leq 1$

$$\left\| \int_{\Delta^l} V_0 e^{-\sigma_0 s^2 D^2} V_1 e^{-\sigma_1 s^2 D^2} \dots V_l e^{-\sigma_l s^2 D^2} d\sigma_1 \dots d\sigma_l \right\| \leq C(q) \|V_0\|_{v_0, t_0} \dots \|V_l\|_{v_l, t_l} s^{-q}$$

for some t_i .

Proof. Using barycentric subdivision, decompose the integral into the sum of integrals over $l + 1$ nonintersecting domains $U_i \subset \Delta^l$ such that $\sigma_i \geq 1/l$ on U_i . Notice that we have the following identity:

$$\begin{aligned} V_0 e^{-\sigma_0 s^2 D^2} V_1 e^{-\sigma_1 s^2 D^2} \dots V_l e^{-\sigma_l s^2 D^2} = \\ (V_0 (1 + D^2)^{-v_0/2}) e^{-\sigma_0 s^2 D^2} ((1 + D^2)^{v_0/2} V_1 (1 + D^2)^{-(v_0/2 + v_1/2)}) e^{-\sigma_1 s^2 D^2} \\ \dots (1 + D^2)^{(v_0/2 + \dots + v_{i-1}/2)} V_i (1 + D^2)^{-(v_0/2 + \dots + v_i/2)} ((1 + D^2)^{v_0/2 + \dots + v_l/2} e^{-\sigma_l s^2 D^2}) \dots \\ ((1 + D^2)^{-(v_{l-1}/2 + v_l/2)} V_{l-1} (1 + D^2)^{v_l/2}) e^{-\sigma_{l-1} s^2 D^2} ((1 + D^2)^{-v_l/2} V_l e^{-\sigma_l s^2 D^2}) e^{-\sigma_l s^2 D^2} \end{aligned}$$

On U_i , using the estimate

$$\|(1 + D^2)^{v_0/2 + \dots + v_l/2} e^{-\sigma_l s^2 D^2}\| \leq c(q) l^{q/2} s^{-q} \text{ when } \sigma \geq \frac{1}{l}, \quad 0 < s \leq 1.$$

we bound the integrand, and obtain the inequality

$$\left\| \int_{U_i} V_0 e^{-\sigma_0 s^2 D^2} V_1 e^{-\sigma_1 s^2 D^2} \dots V_l e^{-\sigma_l s^2 D^2} d\sigma_1 \dots d\sigma_l \right\| \leq c(q) \|V_0\|_{v_0, t_0} \dots \|V_l\|_{v_l, t_l} \frac{l^{q/2} s^{-q}}{l!}$$

The inequality follows since $\frac{l^{q/2}}{l!}$ can be bounded uniformly in l . \square

Choose any p such that $(1/2 \dim N + 1/2) > p > 1/2 \dim N$ and let Q be a properly supported parametrix for $(1 + D^2)^p$, with $1 - Q(1 + D^2)^p = R$ a properly supported smoothing operator. For a zero order pseudodifferential operator V with compactly supported Schwartz kernel let $\mu(V) = \|VQ\|_1 + \|VR\|_1$ where $\|\cdot\|_1$ denotes the trace norm.

Lemma C.2. *Let V_i be as in Lemma C.1. Assume also that V_0 is a zero order pseudodifferential operator with compactly supported Schwartz kernel. Then*

$$\left\| \int_{\Delta^l} V_0 e^{-\sigma_0 s^2 D^2} V_1 e^{-\sigma_1 s^2 D^2} \dots V_l e^{-\sigma_l s^2 D^2} d\sigma_1 \dots d\sigma_l \right\|_1 \leq C'(q) \mu(V_0) \|V_1\|_{v_1, t_1} \dots \|V_l\|_{v_l, t_l} s^{-q - \dim N - 1}$$

for some t_i .

Proof. Notice that

$$\int_{\Delta^l} V_0 e^{-\sigma_0 s^2 D^2} V_1 e^{-\sigma_1 s^2 D^2} \dots V_l e^{-\sigma_l s^2 D^2} d\sigma_1 \dots d\sigma_l = V_0 Q \cdot (1 + D^2)^p P + V_0 R \cdot P$$

where

$$P = \int_{\Delta^l} (1 + D^2)^p e^{-\sigma_0 s^2 D^2} V_1 e^{-\sigma_1 s^2 D^2} \dots V_l e^{-\sigma_l s^2 D^2} d\sigma_1 \dots d\sigma_l$$

Now $(1 + D^2)^p P$ and P are bounded operators, whose norms can be estimated by Lemma C.1. \square

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